Short spinning strings

and $AdS_5 \times S^5$ spectrum

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70-s: origin of

• string theory

• supersymmetric field theories in 2d, 4d, 10d

[Lars Brink's talk]

but also of

• conformal field theory in 4d

[Polyakov, Ferrara et al, ...]

• integrability of 2d sigma models

[Pohlmeyer, Luescher, Polyakov, Zamolodchikovs, ...]

and also of planar limit ($N \to \infty, \ \lambda = g^2_{\rm YM} N$=fixed) ['t Hooft]

$\mathcal{N} = 4$ SYM: remarkable relation to all of these ideas

Relation of SYM to string theory? known since 70's / early 80's:

[Scherk et al; Brink, Green, Schwarz]

On-shell scattering amplitudes of $\mathcal{N} = 4$ SYM may be found from $\alpha' \to 0$ limit of superstring amplitudes in flat 10d space $R^{1,3} \times T^6$

But what about correlation functions of composite operators
= natural observables in a CFT ?
can one compute 2-point (spectrum of dimensions)
and 3-point (OPE coefficients) correlation functions of
conformal primary operators
[e.g. $\langle \mathrm{Tr} F^2_{mn}(x_1) ... \mathrm{Tr} F^2_{pq}(x_n) \rangle$]
from 10d flat-space string theory?!
not in an obvious or useful way (need an off-shell extension,...)

the idea that one can use string theory to solve $\mathcal{N} = 4$ SYM
as a CFT using string theory would sound wild in 70's ...
... especially if this string theory is in curved space
and with finite tension ...

Maximally symmetric case of gauge-string duality:

[Maldacena, Polyakov, Klebanov, Witten,...]

planar $\mathcal{N}=4$ super Yang-Mills $\leftrightarrow$ free $AdS_5 \times S^5$ superstring

closed string states on $R\times S^1 \leftrightarrow$ gauge-inv. SYM states on $R\times S^3$

marginal str. vertex ops on $R^2 \leftrightarrow$ conf. primary SYM ops on R^4

Remarkably, correlators of $AdS_5 \times S^5$ string vertex operators

– analogs of S-matrix elements in flat 10d space —

are (expected to be) dual to correlators of

conformal operators of planar $\mathcal{N}=4$ SYM

In particular, relation of 2-point functions means that

spectrum of $AdS_5 \times S^5$ string energies

$\leftrightarrow$ spectrum of dimensions of SYM primary operators

Then the spectrum of $\mathcal{N}=4$ SYM conformal dimensions $\Delta(\lambda)$

should be possible to describe by 2d $AdS_5\times S^5$ superstring sigma

Integrability:

allows "in principle" to solve the problem of spectrum

enormous progress in the last 10 years

Some key inputs: [not a proper review!]

- SYM action + perturbation theory ($\lambda \ll 1$)
- $AdS_5 \times S^5$ GS superstring action + α'-expansion ($\sqrt{\lambda} \gg 1$)
- classical integrability of $AdS_5 \times S^5$ GS action
  [Bena, Polchinski, Roiban]
- perturbative integrability of SYM spectral problem:
  (1-loop, 2-loop, ...) dilatation operator = spin chain Hamiltonian
  [Minahan, Zarembo; Beisert, Staudacher, ...]
- guidance from large-charge limits: BMN, GKP, FT

Assume that integrability extends to all orders on both sides

- construct interpolating Bethe ansatz guided by general principles, symmetries and data from both weak+strong coupling
- check consistency of its predictions

I. Spectrum of "long" operators / "semiclassical" string states
determined by Asymptotic Bethe Ansatz (2002-2007)

- its final [Beisert-Eden-Staudacher] form found by intricate superposition of data from $\lambda \ll 1$ gauge theory (spin chain, BA,...) and perturbative string theory (classical and 1-loop phase, BMN), symmetries (S-matrix), assumption of exact integrability
- consequences checked against available gauge and string data

Key example: cusp anomalous dimension – of $\mathrm{Tr}(\Phi D^S \Phi)$

$$\Delta = S + 2 + f(\lambda) \ln S + \ldots , \qquad S \gg 1$$

$$f_{\lambda \ll 1} = \frac{\lambda}{2\pi^2}\Big[1 - \frac{\lambda}{48} + \frac{11\lambda^2}{45 \cdot 2^8} - (\frac{73}{630} + \frac{4\zeta_3^2}{\pi^6})\frac{\lambda^3}{2^7} + \ldots\Big]$$

$$f_{\lambda \gg 1} = \frac{\sqrt{\lambda}}{\pi}\Big[1 - \frac{3\ln 2}{\sqrt{\lambda}} - \frac{K}{(\sqrt{\lambda})^2} - \ldots\Big]$$

$\zeta_k = \zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}, \quad K = \beta(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915\ldots$

from 2-loop string sigma-model integrals [Roiban,Tirziu,AT]

exact integral eq. [Basso, Korchemsky, Kotanski]: any order term

II. Spectrum of "short" operators = quantum string states

Thermodynamic Bethe Ansatz (2005-...)

- reconstructed from ABA using solely methods/intuition of 2-d integrable QFT, i.e. inspired by string-theory side
- highly non-trivial construction – lack of 2-d Lorentz invariance in standard BMN-vacuum-adapted l.c. gauge
- in few cases ABA "improved" by Luscher corrections is enough: 4- and 5-loop Konishi dim, 4-loop dim. of twist 2 operator
- complicated set of integral equations in need of simplification; so far predictions extracted only numerically starting from weak coupling and interpolating to larger λ
- need more data to check predictions at $\lambda \ll 1$ and $\lambda \gg 1$, i.e. against perturbative gauge-theory and string-theory data

Key example:

dimension $\Delta = 2 + \gamma(\lambda)$ of Konishi operator $\mathrm{Tr}(\bar{\Phi}_i \Phi_i)$

$$\gamma(\lambda \ll 1) = \frac{12\lambda}{(4\pi)^2}\Big[1 - \frac{4\lambda}{(4\pi)^2} + \frac{28\lambda^2}{(4\pi)^4} - \big(208 - 48\zeta_3 + 120\zeta_5\big)\frac{\lambda^3}{(4\pi)^6} + 8\big(158 + 72\zeta_3 - 54\zeta_3^2 - 90\zeta_5 + 315\zeta_7\big)\frac{\lambda^4}{(4\pi)^8} + ...\Big]$$

5-loop result first found using integrability [Banjok,Janik]
confirmed by more standard methods [Velizhanin; Eden et al 12]

Suppose one can sum up (convergent) $\lambda \ll 1$ expansion
and then re-expand at $\lambda \gg 1$

What one should expect to get for $\gamma(\lambda \gg 1)$?

Duality to string theory predicts the structure
of strong-coupling expansion:
leading term – near-flat-space expansion for fixed quant. numbers
[Gubser, Klebanov, Polyakov 98]

$$\Delta = \sqrt{2N\sqrt{\lambda}} + ...$$

Subleading terms: $\alpha' = \frac{1}{\sqrt{\lambda}}$ expansion of 2d anom. dimensions
of corresponding vertex operators [Roiban, AT 09] ($N = 2$)

$$\begin{aligned}\gamma(\lambda \gg 1) &= 2\sqrt[4]{\lambda} + \frac{b_1}{\sqrt[4]{\lambda}} + \frac{b_2}{(\sqrt[4]{\lambda})^3} + \frac{b_3}{(\sqrt[4]{\lambda})^{5/2}} + ... \\ &= 2\sqrt[4]{\lambda}\Big[1 + \frac{b_1}{2\sqrt{\lambda}} + \frac{b_2}{2(\sqrt{\lambda})^2} + \frac{b_3}{2(\sqrt{\lambda})^3} + ...\Big]\end{aligned}$$

Values of b_k from string theory? From TBA?

Dimensions of "short" SYM operators

= energies of quantum string states

find leading $\alpha' = \frac{1}{\sqrt{\lambda}}$ corrections to energy of

"lightest" massive string states on first massive string level

dual to operators in Konishi multiplet in SYM theory

– compare with predictions of TBA approach

important to check integrability-based approach

which involves subtle assumptions

directly against perturbative string sigma model

[similar checks were done, e.g., in standard $O(n)$ models]

TBA results:

start at weak coupling for sl(2) Konishi descendant $\mathrm{Tr}(\Phi D^2 \Phi)$
use TBA to find $\Delta(\lambda)$ numerically;
match to expected form of strong-coupling expansion to extract b_k
[Gromov, Kazakov, Vieira 09; Frolov 10, 12]

$$b_1 \approx 1.988\,, \qquad b_2 \approx -3.07$$

Compare to string theory:

One can find b_k using semiclassical "short string" expansion
[Roiban, AT 09, 11; Gromov, Serban, Shenderovich, Volin 11]

$$b_1 = 2\,, \qquad b_2 = a - 3\zeta_3$$

rational a is found [Gromov, Valatka 11] using also "2-loop"
coefficient in exact slope function ($E^2 = h(\lambda) S$) [Basso 11]

$$b_2 = \frac{1}{2} - 3\zeta_3 \approx -3.106...$$

Remarkable agreement with TBA - check of quantum integrability

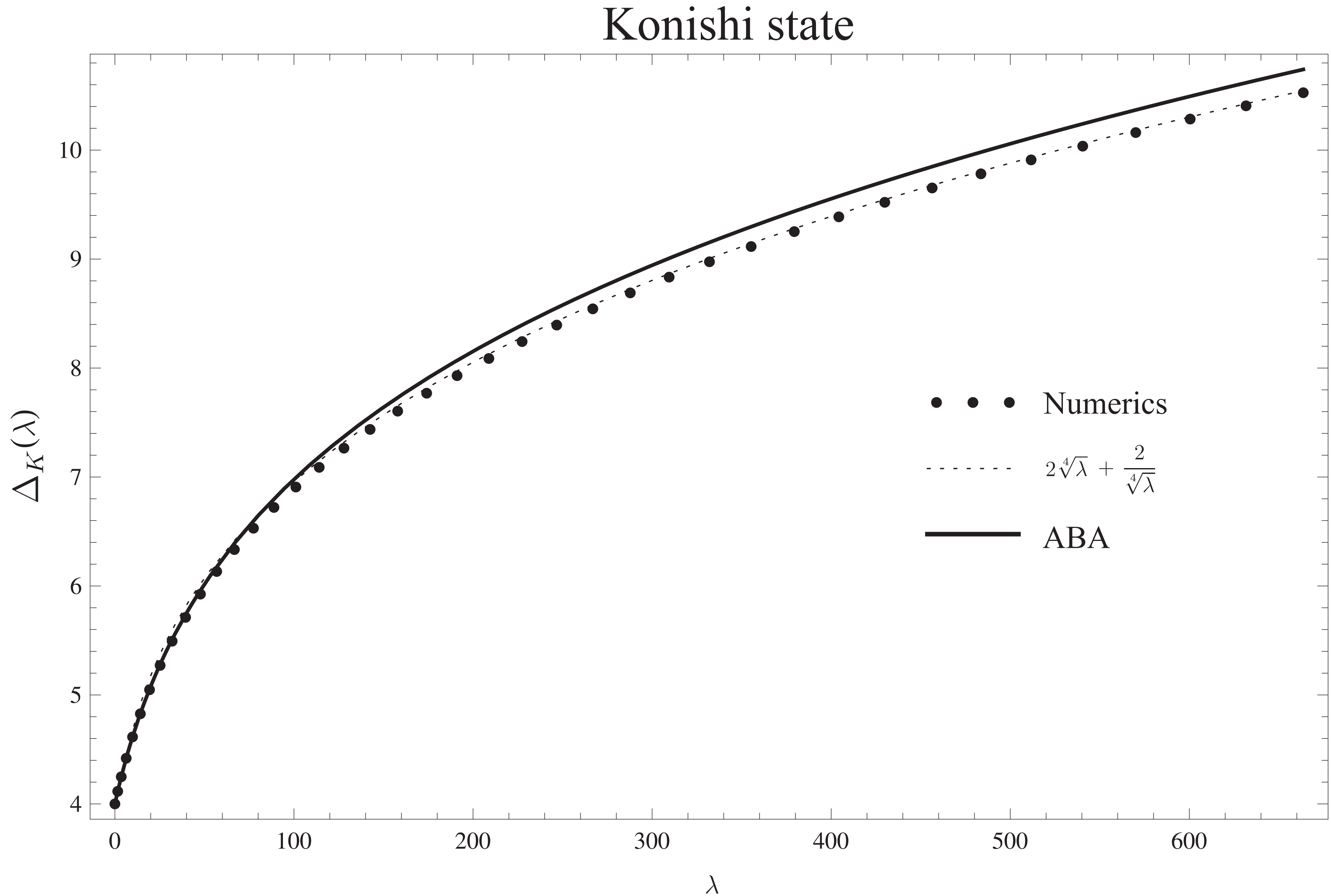

Figure 1: Plot from Gromov, Kazakov, Vieira

Recent work on string side: [BGMRT 12]

• highest transcendentality terms in b_k
are $\sim \zeta_{2k-1}$ and have 1-loop origin, e.g.,

$$b_3 = a_1 + a_2\zeta_3 + a_3\zeta_5 \ , \qquad a_3 = \frac{15}{2}$$

rational a_1 receives contribution from 3 loops; a_2 from
2-loops, etc.; $b_4 \sim \zeta_7 + ...$, etc.

• supermultiplet structure: universality of coefficients in E
for string states with spins in different $AdS_5 \times S^5$ directions:
dual operators from Konishi multiplet have same energy
(up to constant shift depending on position in the multiplet)

• states on leading Regge trajectory:
general structure of dependence of energy on
string tension $\sqrt{\lambda}$, string level (spin) and S^5 orbital momentum J

Some open questions:

- Analytic form of strong-coupling expansion from TBA?

- only ζ_k coefficients in $\Delta(\lambda)$ in both
weak and strong coupling expansions
or other transcendental constants may also appear?
(cf. cusp anomalous dimension)
2-loop string computation may shed light on this ...

- Asymptotic form of strong coupling expansion:
interpretation of possible $e^{-a\sqrt{\lambda}}$ corrections?

- Energies of other quantum states: similar behavior?

Konishi multiplet:

long multiplet related to singlet $[0,0,0]_{(0,0)}$ by susy

$[J_2 - J_3, J_1 - J_2, J_2 + J_3]_{(s_L, s_R)}$
$s_{L,R} = \frac{1}{2}(S_1 \pm S_2)$

$SO(6)$ (J_1, J_2, J_3) and $SO(4)$ (S_1, S_2) labels
of $SO(2,4) \times SO(6)$ global symmetry
[Andreanopoli,Ferrara 98; Bianchi,Morales,Samtleben 03]

$\Delta = \Delta_0 + \gamma(\lambda), \qquad \Delta_0 = 2, \frac{5}{2}, 3, ..., 10$
same anomalous dimension γ for all members

singlet eigen-state of anom. dim. matrix with lowest eigenvalue

Examples of gauge-theory operators in Konishi multiplet:

$[0,0,0]_{(0,0)}$:

$\mathrm{Tr}(\bar{\Phi}_i \Phi_i), \quad i = 1,2,3, \qquad \Delta_0 = 2$

$[2,0,2]_{(0,0)}$:

$\mathrm{Tr}([\Phi_1, \Phi_2]^2)$ in $su(2)$ sector, $\Delta_0 = 4$

$[0,2,0]_{(1,1)}$:

$\mathrm{Tr}(\Phi_1 D^2 \Phi_1)$ in $sl(2)$ sector, $\Delta_0 = 4$

Δ_0	
2	$[0,0,0]_{(0,0)}$
$\frac{5}{2}$	$[0,0,1]_{(0,\frac{1}{2})} + [1,0,0]_{(\frac{1}{2},0)}$
3	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})} + [0,0,2]_{(0,0)} + [0,1,0]_{(0,1)+(1,0)} + [1,0,1]_{(\frac{1}{2},\frac{1}{2})} + [2,0,0]_{(0,0)}$
$\frac{7}{2}$	$[0,0,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)+(\frac{3}{2},0)} + [0,1,1]_{(0,\frac{1}{2})+(1,\frac{1}{2})} + [1,0,0]_{(0,\frac{1}{2})+(0,\frac{3}{2})+(1,\frac{1}{2})} + [1,0,2]_{(\frac{1}{2},0)}$ $+[1,1,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)} + [2,0,1]_{(0,\frac{1}{2})}$
4	$[0,0,0]_{(0,0)+(0,2)+(1,1)+(2,0)} + [0,0,2]_{(\frac{1}{2},\frac{1}{2})+(\frac{3}{2},\frac{1}{2})} + [0,1,0]_{2(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})} + [2,0,2]_{(0,0)}$ $+[0,1,2]_{(1,0)} + [0,2,0]_{2(0,0)+(1,1)} + [1,0,1]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [1,1,1]_{2(\frac{1}{2},\frac{1}{2})} + [2,0,0]$
6	$[0,0,0]_{3(0,0)+3(1,1)+(2,2)} + [0,0,2]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})} + [0,1,0]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})+}$ $+[0,1,2]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [0,2,0]_{3(0,0)+(0,1)+(0,2)+(1,0)+3(1,1)+(2,0)} + [0,2,2]_{(\frac{1}{2},\frac{1}{2})}$ $+[0,3,0]_{2(\frac{1}{2},\frac{1}{2})} + [0,4,0]_{(0,0)} + [1,0,1]_{(0,0)+3(0,1)+3(1,0)+4(1,1)+(1,2)+(2,1)} + [1,0,3]_{(\frac{1}{2},\frac{1}{2})} + [0$ $+[1,1,1]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})} + [1,2,1]_{(0,0)+(0,1)+(1,0)} + [2,0,0]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})}$ $+[2,0,2]_{(0,0)+(1,1)} + [2,1,0]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [2,2,0]_{(\frac{1}{2},\frac{1}{2})} + [3,0,1]_{(\frac{1}{2},\frac{1}{2})} + [4,0,0]$
$\frac{17}{2}$	$[0,0,1]_{(0,\frac{1}{2})+(0,\frac{3}{2})+(1,\frac{1}{2})} + [0,1,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)} + [1,0,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)+(\frac{3}{2},0)} + [1,0,2]_{(0,\frac{1}{2})}$ $+[1,1,0]_{(0,\frac{1}{2})+(1,\frac{1}{2})} + [2,0,1]_{(\frac{1}{2},0)}$
9	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})} + [0,0,2]_{(0,0)} + [0,1,0]_{(0,1)+(1,0)} + [1,0,1]_{(\frac{1}{2},\frac{1}{2})} + [2,0,0]_{(0,0)}$
$\frac{19}{2}$	$[0,0,1]_{(\frac{1}{2},0)} + [1,0,0]_{(0,\frac{1}{2})}$
10	$[0,0,0]_{(0,0)}$

Table 1: Long Konishi multiplet (a bit cut off)

Comparison between gauge and string theory states:

$\lambda \ll 1:$ gauge-theory operators built out of free fields, canonical dim. Δ_0 determines operators that can mix

$\lambda \gg 1$: in near-flat-space expansion string states built out of free oscillators, level N determines states that can mix

(i) relate states with same global charges

(ii) assume direct interpolation (no "level crossing") for states with same quantum numbers as λ changes from small to large values

Gauge-string duality:

Konishi operator dual to
"lightest" among massive $AdS_5 \times S^5$ string states

large $\sqrt{\lambda} = \frac{\mathrm{R}^2}{\alpha'}$:
"short" strings probe near-flat limit of $AdS_5 \times S^5$

members of supermultiplet:
strings with spins/oscillators in different $AdS_5 \times S^5$ directions

Flat space case:

$m^2 = \frac{2N}{\alpha'}$

$N = 0$: massless IIB supergravity (BPS) level
l.c. vacuum $|0\rangle$: $(8+8)^2 = 256$ states

$N = 2$: first massive level (many states, highly degenerate)
$[(a^i_{-1} + S^a_{-1})|0\rangle]^2 = [(8+8) \times (8+8)]^2$
in $SO(9)$ reps:
$([2,0,0,0] + [0,0,1,0] + [1,0,0,1])^2 = (44 + 84 + 128)^2$
e.g. $44 \times 44 = 1 + 36 + 44 + 450 + 495 + 910$
$84 \times 84 = 1 + 36 + 44 + 84 + 126 + 495 + 594 + 924 + 1980 + 2772$

switching on $AdS_5 \times S^5$ background fields lifts degeneracy
states with "lightest mass" at first excited string level
should correspond to Konishi multiplet

String spectrum in $AdS_5 \times S^5$:

long multiplets of $PSU(2,2|4)$
highest weight states:
$[J_2 - J_3, J_1 - J_2, J_2 + J_3]_{(s_L, s_R)}, \quad s_{L,R} = \frac{1}{2}(S_1 \pm S_2)$

Flat-space string spectrum can be re-organized
in multiplets of $SO(2,4) \times SO(6) \subset PSU(2,2|4)$
[Bianchi, Morales, Samtleben 03; Beisert et al 03]
$SO(4) \times SO(5) \subset SO(9)$ rep.
lifted to $SO(4) \times SO(6)$ rep. of $SO(2,4) \times SO(6)$

Konishi long multiplet
$\mathcal{K} = (1 + Q + Q \wedge Q + ...)[0,0,0]_{(0,0)}$
determines the "floor" of 1-st excited string level
$\sum_{J=0}^{\infty} [0, J, 0]_{(0,0)} \times \mathcal{K}$

Spins: S_1, S_2 in AdS_5; (J_1, J_2) in S^5
orbital momentum $J = J_3$ in S^5

Examples:

- folded string with spin S_1 and momentum J:

$S_1 = J = 2 \rightarrow \quad [0, 2, 0]_{(1,1)}, \quad \Delta_0 = 4$

- folded string with spin J_1 and momentum J:

$J_1 = J = 2 \rightarrow \quad [2, 0, 2]_{(0,0)}, \quad \Delta_0 = 4$

- circular string with spins $J_1 = J_2$ and momentum J:

$J_1 = J_2 = 1, J = 2 \rightarrow \quad [0, 1, 2]_{(0,0)}, \quad \Delta_0 = 6$

- circular string with spins $S_1 = S_2$ and momentum J:

$S_1 = S_2 = 1, J = 2 \rightarrow \quad [0, 1, 2]_{(0,0)}, \quad \Delta_0 = 6$

- circular string with spins $S_1 = J_1$ and momentum J:

$S_1 = J_1 = 1, J = 2 \rightarrow \quad [1, 1, 1]_{(\frac{1}{2},\frac{1}{2})}, \quad \Delta_0 = 6$

Direct approaches to computation of quantum string energies:
(not relying on integrability)
(i) vertex operator approach:
use $AdS_5 \times S^5$ string sigma model perturbation theory to find leading terms in 2d anomalous dimension of corresponding vertex operators and impose marginality condition
[Polyakov 01; AT 03; pure spinors: Mazzucato,Vallilo 11]
(ii) "light-cone" gauge approach:
start with AdS light-cone gauge $AdS_5 \times S^5$ string action and compute corrections to energy of corresponding flat-space oscillator string state [Metsaev, Thorn, AT 00]

both yet to be developed in detail;
in practice, will be guided by vertex operator approach but use indirect but easier to implement
"semiclassical" approach: "short string" limit of semiclassical expansion [Tirziu, AT 08; Roiban, AT 09, 11]

Target space perspective:

string in flat space: $p^2 = m^2 = \frac{2N}{\alpha'}$

e.g. leading Regge trajectory $(\partial x \bar{\partial} x)^{S/2} e^{ipx}, \quad N = S$

spectrum in (weakly) curved background:

solve marginality (1,1) conditions on vertex operators

e.g. 2d anomalous dimension operator $\widehat{\gamma}$ on scalar $T(x)$

differential operator in target space

from β-function for the corresponding perturbation

$$I = \frac{1}{4\pi\alpha'} \int d^2 z \Big[G_{mn}(x) \partial x^m \bar{\partial} x^n + T(x) \Big]$$

$$\beta_T = -2T - \tfrac{\alpha'}{2}\, \widehat{\gamma}\, T + O(T^2)$$

$$\widehat{\gamma} = \Omega^{mn} D_m D_n + ... + \Omega^{m...k} D_m ... D_k + ...$$

$$\Omega^{mn} = G^{mn} + O(\alpha'^3), \qquad\qquad \Omega^{\cdots\cdot} \sim \alpha'^n R^p_{\cdots\cdot}$$

Solving $-\widehat{\gamma}\, T + m^2 T = 0 \qquad (m^2 = -\frac{4}{\alpha'})$

same as diagonalizing $\widehat{\gamma}$

Massive string states in curved background:

$$\int d^D x \sqrt{g}\, \Big[\Phi_{...}(-D^2 + m^2 + X)\Phi_{...} + ...\Big]$$

$$m^2 = \frac{2N}{\alpha'}\ , \qquad X = R_{....} + O(\alpha')$$

case of $AdS_5 \times S^5$ background

$$R_{mn} - \frac{1}{96}(F_5 F_5)_{mn} = 0, \qquad R = 0\ , \qquad F_5^2 = 0$$

Find leading-order term in X ...

leading α' correction to scalar string state mass is 0 (?!)

$$[-D^2 + m^2 + O(\frac{1}{\sqrt{\lambda}})]\Phi\ = 0$$

$$\Delta = 2 + \sqrt{2N + 4 + O(\frac{1}{\sqrt{\lambda}})}$$

$$\Delta_{N=2} = 2 + 2\sqrt[4]{\lambda}\, \Big[1 + \frac{1}{2\sqrt{\lambda}} + O(\frac{1}{(\sqrt{\lambda})^2})\Big]$$

Too naive: various subtleties (10d scalar vs singlet state, mixing)

What is found for non-singlet (susy descendant) Konishi states?

Vertex operator approach

calculate 2d anomalous dimensions from "first principles"–
superstring theory in $AdS_5 \times S^5$:

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \Big[\partial Y_p \bar{\partial} Y^p + \partial X_k \bar{\partial} X_k + \text{fermions} \Big]$$

$$-Y_0^2 - Y_5^2 + Y_1^2 + ... + Y_4^2 = -1 \,, \qquad X_1^2 + ... + X_6^2 = 1$$

construct marginal (1,1) operators in terms of Y_p and X_k
e.g. vertex operator for dilaton (in NSR framework)

$$\text{V}_J = \, (Y_+)^{-\Delta} \, (X_x)^J \, \Big[\partial Y_p \bar{\partial} Y^p + \partial X_k \bar{\partial} X_k + \text{fermions} \Big]$$

$$Y_+ \equiv Y_0 + iY_5 = z + z^{-1} x_m x_m \sim e^{it}$$
$$X_x \equiv X_1 + iX_2 \sim e^{i\varphi}$$

$$2 = 2 + \frac{1}{2\sqrt{\lambda}} [\Delta(\Delta - 4) - J(J+4)] + O(\tfrac{1}{(\sqrt{\lambda})^2})$$

i.e. $\Delta = 4 + J$ (BPS)

Vertex operators = eigenstates of 2d anomalous dimension matrix

particular linear combinations like

$$V = f_{k_1 ... k_\ell m_1 ... m_{2s}} X_{k_1} ... X_{k_\ell} \partial X_{m_1} \bar\partial X_{m_2} ... \partial X_{m_{2s-1}} \bar\partial X_{m_{2s}}$$

their renormalization studied in $O(n)$ sigma model [Wegner 90]

simplest case: $f_{k_1 ... k_\ell} X_{k_1} ... X_{k_\ell}$ with traceless $f_{k_1 ... k_\ell}$

highest-weight rep $V_J = (X_x)^J$, $\widehat{\gamma} = 2 - \frac{1}{2\sqrt{\lambda}} J(J+4) + ...$

Higher massive states:

Flat space: bosonic string state on leading Regge trajectory

$$\mathrm{V}_S = e^{-iEt} \big(\partial \mathrm{x} \bar\partial \mathrm{x}\big)^{S/2}, \quad \mathrm{x} = x_1 + i x_2 \,, \qquad \alpha' E^2 = 2(S-2)$$

$AdS_5 \times S^5$: candidates for operators on leading Regge trajectory:

$$\mathrm{V}_J = (Y_+)^{-\Delta} \big(\partial X_x \bar\partial X_x\big)^{J/2} \,, \qquad X_x \equiv X_1 + i X_2$$

$$\mathrm{V}_S = (Y_+)^{-\Delta} \big(\partial Y_u \bar\partial Y_u\big)^{S/2} \,, \qquad Y_u \equiv Y_1 + i Y_2$$

+ fermionic terms

+ $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ terms from diagonalization of anom. dim. op.

– mixing with operators with same charges and dimension.

In general $\left(\partial X_x \bar{\partial} X_x\right)^{J/2}$ mixes with singlets

$$(X_x)^{2p+2q} (\partial X_x)^{\frac{1}{2}J'-2p} (\bar{\partial} X_x)^{\frac{1}{2}J'-2q} (\partial X_m \partial X_m)^p (\bar{\partial} X_k \partial X_k)^q$$

Example of higher-level scalar/singlet operator:

$$Y_+^{-\Delta} \Big[(\partial X_k \bar{\partial} X_k)^r + ... \Big] , \qquad N = 2(r-1)$$

Marginality condition:

[cf. Kravtsov, Lerner, Yudson 89; Castilla, Chakravarty 96]

$$0 = 2(r-1) - \frac{1}{2\sqrt{\lambda}} \Big[\Delta(\Delta - 4) + 2r(r-1) \Big] - \frac{1}{(\sqrt{\lambda})^2} \Big[\tfrac{2}{3} r(r-1)(r - \tfrac{7}{2}) + 4r \Big] + ...$$

$r = 1$: ground level– fermions should make $r = 1$ zero of $\widehat{\gamma}$

$r=2$: excited level – analog of singlet Konishi state $\Delta_0=2$

$$\Delta(\Delta-4)=4\sqrt{\lambda}-4+O(\tfrac{1}{\sqrt{\lambda}})\ ,$$
$$\Delta-\Delta_0=2\sqrt[4]{\lambda}\Big[1+0\times\tfrac{1}{\sqrt{\lambda}}+O(\tfrac{1}{(\sqrt{\lambda})^2})\Big]$$

fermionic contributions change subleading coefficients

Bosonic operators with momentum J and spin $J_1\equiv J'$ in S^5:

$$V_{J,J'}=Y_+^{-\Delta}\sum_{k,m=0}^{J'/2}c_{km}M_{km}$$
$$M_{km}\equiv X_y^{J-k-m}X_x^{k+m}(\partial X_y)^k(\partial X_x)^{\frac{1}{2}J'-k}(\bar\partial X_y)^m(\bar\partial X_x)^{\frac{1}{2}J'-m}$$

highest and lowest eigen-values of 1-loop anom. dim. matrix

$$\widehat{\gamma}_{\rm min}=2-J'+\frac{1}{2\sqrt{\lambda}}\Big[\Delta(\Delta-4)-J(J+4)-\tfrac{1}{2}J'(J'+10)-2JJ'\Big]+...$$
$$\widehat{\gamma}_{\rm max}=2-J'+\frac{1}{2\sqrt{\lambda}}\Big[\Delta(\Delta-4)-J(J+4)-\tfrac{1}{2}J'(J'+6)\Big]+...$$

fermionic contributions change terms linear in J'

How to take fermionic contributions into account?

(i) compute energies of semiclassical string states in $\frac{1}{\sqrt{\lambda}}$ expansion
using full $AdS_5 \times S^5$ Green-Schwarz action

(ii) compare to structure of $E = \Delta$
expected from marginality condition

(iii) determine unknown coefficients in $E(\sqrt{\lambda})$

General structure of dimension/energy $\Delta = E$

marginality condition – condition on quantum numbers Q_i

$Q = (E(\lambda), S_1, S_2; J_1, J_2, J_3; ...)$; $\quad N = \sum_i a_i Q_i$ = level

$$0 = 2N + \frac{1}{\sqrt{\lambda}}\Big(\sum_{i,j} c_{ij} Q_i Q_j + \sum_i c_i Q_i\Big)$$
$$+\frac{1}{(\sqrt{\lambda})^2}\Big(\sum_{i,j,k} c_{ijk} Q_i Q_j Q_k + \sum_{i,j} c'_{ij} Q_i Q_j + \sum_i c'_i Q_i\Big) + ...$$

- highest power of Q at n-loop order $\frac{1}{(\sqrt{\lambda})^n} Q^{n+2}$
- dependence on “momenta” E and $J = J_3$ is special

  e.g., they do not enter N
- terms $\frac{1}{(\sqrt{\lambda})^n} E^m$ are scheme-dependent (cf.$(\alpha' D^2)^n$ terms in $\widehat{\gamma}$)

choose susy-preserving scheme where $E = J$ is the only solution in BPS case – only E^2 terms are then left (ignore const shifts)

- terms $\frac{1}{(\sqrt{\lambda})^n} E^2 Q^k$ can be traded for E-independent terms

in perturbative solution for E

States on "leading Regge trajectory":

(max spin for given E in flat limit)

$Q = (E, J; N)$, N= spin component

marginality condition:

$$0 = 2N + \frac{1}{\sqrt{\lambda}}\Big(- E^2 + J^2 + n_{02}N^2 + n_{11}N\Big) + \frac{1}{(\sqrt{\lambda})^2}\Big(n_{01}J^2N + n_{03}N^3 + n_{12}N^2 + n_{21}N\Big) + O(\frac{1}{(\sqrt{\lambda})^3})$$

solution for E^2 takes form [Roiban, AT 09, 11; BGMRT 12]

$$E^2 = 2\sqrt{\lambda}N + J^2 + n_{02}N^2 + n_{11}N + \frac{1}{\sqrt{\lambda}}(n_{01}J^2N + n_{03}N^3 + n_{12}N^2 + n_{21}N) + \frac{1}{(\sqrt{\lambda})^2}(\widetilde{n}_{11}J^2N + \widetilde{n}_{02}J^2N^2 + n_{04}N^4 + n_{13}N^3 + n_{22}N^2 + n_{31}N) + \frac{1}{(\sqrt{\lambda})^3}(\widetilde{n}_{01}J^4N + \widetilde{n}_{21}J^2N + \widetilde{n}_{12}J^2N^2 + n_{05}N^5 + ...) + O(\frac{1}{(\sqrt{\lambda})^4})$$

J^kN^m terms: non-decoupling of c.o.m.

Expanding in large $\sqrt{\lambda}$ for fixed N, J

$$E = \sqrt{2\sqrt{\lambda}N}\Big[1 + \frac{A_1}{\sqrt{\lambda}} + \frac{A_2}{(\sqrt{\lambda})^2} + O(\frac{1}{(\sqrt{\lambda})^3})\Big]$$
$$A_1 = \frac{1}{4N}J^2 + \frac{1}{4}(n_{02}N + n_{11})$$
$$A_2 = -\frac{1}{2}A_1^2 + \frac{1}{4}(n_{01}J^2 + n_{03}N^2 + n_{12}N + n_{21})$$

Gives strong-coupling expansion of dimension
of corresponding gauge-theory operator

States on 1-st excited superstring level: $N = 2$
Konishi multiplet states: $N = 2$, $J = 2$

$$E = \sqrt[4]{\lambda}\Big[2 + \frac{b_1}{\sqrt{\lambda}} + \frac{b_2}{(\sqrt{\lambda})^2} + O(\frac{1}{(\sqrt{\lambda})^3})\Big]$$
$$b_1 = 1 + n_{02} + \tfrac{1}{2}n_{11}$$
$$b_2 = -4b_1^2 + 2n_{01} + 2n_{03} + n_{12} + \tfrac{1}{2}n_{21}$$

To find coefficients n_{km}

use semiclassical "short string" (small spin) expansion:

• start with solitonic string carrying same charges

as vertex operator representing particular quantum string state

• perform semiclassical expansion: $\sqrt{\lambda} \gg 1$

for fixed classical parameters $(\mathcal{N}, \mathcal{J})$

$\mathcal{N} = \frac{1}{\sqrt{\lambda}} N, \quad \mathcal{J} = \frac{1}{\sqrt{\lambda}} J$

• expand E in small values of $\mathcal{N}, \mathcal{J}$

• re-interpret the resulting E in terms of N, J

and read off n_{km}

Key point: limit $\mathcal{N} = \frac{N}{\sqrt{\lambda}} \to 0, \ \ \mathcal{J} = \frac{J}{\sqrt{\lambda}} \to 0$

corresponds to $\sqrt{\lambda} \gg 1$ for fixed values of quantum charges N, J

Rewrite E^2 in terms of $\mathcal{N}, \mathcal{J}$:

$$
\begin{aligned}
(\frac{E}{\sqrt{\lambda}})^2 &= (2\mathcal{N} + \mathcal{J}^2 + n_{01}\mathcal{J}^2\mathcal{N} + n_{02}\mathcal{N}^2 + n_{03}\mathcal{N}^3 + \widetilde{n}_{01}\mathcal{J}^4\mathcal{N} + \widetilde{n}_{02}\mathcal{J}^2\mathcal{N}^2 + ...) \\
&+ \frac{1}{\sqrt{\lambda}}(n_{11}\mathcal{N} + \widetilde{n}_{11}\mathcal{J}^2\mathcal{N} + n_{12}\mathcal{N}^2 + \widetilde{n}_{12}\mathcal{J}^2\mathcal{N}^2 + n_{13}\mathcal{N}^3 + ...) \\
&+ \frac{1}{(\sqrt{\lambda})^2}(n_{21}\mathcal{N} + \widetilde{n}_{21}\mathcal{J}^2\mathcal{N} + n_{22}\mathcal{N}^2 + ...) + O(\tfrac{1}{(\sqrt{\lambda})^3}) ,
\end{aligned}
$$

• can now interpret n_{km} as semiclassical k-loop contribution to $\mathcal{N}^m$ term in E: 1-st line is classical energy, 2nd is 1-loop, etc.

• semiclassical loop expansion ($\sqrt{\lambda} \gg 1$ for fixed $\mathcal{N}, \mathcal{J}$) different from first loop expansion –in 2d anomalous dimension– ($\sqrt{\lambda} \gg 1$ for fixed N, J) but they contain same coefficients

• each loop term in first expansion is polynomial in charges but in semiclassical expansion each term may contain infinite series in small $\mathcal{J}, \mathcal{N}$ expansion

Semiclassical expansion of E^2 organized as expansion in small $\mathcal{N}$ formally looks like series in powers of N:

$$E^2 = J^2 + \ h_1(\lambda, J)\ N + \ h_2(\lambda, J)\ N^2 + \ h_3(\lambda, J)\ N^3 + ...$$

for fixed J and large λ

$$h_1 = 2\sqrt{\lambda} + n_{11} + \frac{n_{21}}{\sqrt{\lambda}} + \frac{n_{31}}{(\sqrt{\lambda})^2} + ... + J^2\Big(\frac{n_{01}}{\sqrt{\lambda}} + \frac{\widetilde{n}_{11}}{(\sqrt{\lambda})^2} + ...\Big) + ...$$

$$h_2 = n_{02} + \frac{n_{12}}{\sqrt{\lambda}} + ... \ , \qquad\qquad h_3 = \frac{n_{03}}{\sqrt{\lambda}} + ...$$

exact result for “slope” h_1 for AdS_5 folded spinning string state ($N = S$) from Bethe Ansatz [Basso 11]

$$\begin{aligned} h_1(\lambda, J) &= 2\sqrt{\lambda}\frac{d}{d\sqrt{\lambda}} \ln I_J(\sqrt{\lambda}) \\ &= 2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2} - \frac{1}{1+\mathcal{J}^2} - \frac{\frac{1}{4} - \mathcal{J}^2}{\sqrt{\lambda}(1+\mathcal{J}^2)^{5/2}} + ... \\ &= 2\sqrt{\lambda + J^2} - \frac{\lambda}{\lambda + J^2} - \frac{\lambda(\frac{1}{4}\lambda - J^2)}{(\lambda + J^2)^{5/2}} + ... \end{aligned}$$

[Can one find h_1 by direct summation of 4d or 2d Feynmann diagrams (or via localization) ?]

But one needs to know also coefficients in $h_2, h_3, ...$ (much more non-trivial, depend on wrapping corrections)

Strategy: consider examples of "small" semiclassical string states corresponding to quantum string states with angular momentum J and few oscillator modes excited (carrying spin)

- start with classical string solutions in flat space representing states on leading Regge trajectory
- find the corresponding solutions in $AdS_5 \times S^5$
- find 1-loop correction to their E
- expand E in $\mathcal{N} = \frac{N}{\sqrt{\lambda}} \to 0$ – interpolate result to finite N
- find the coefficients n_{km}
- check universality of E for $N = 2$ (implied by susy)

Two basic classes of examples (N= spin, J= orbital momentum):

- circular string with 2 spins in two orthogonal planes
- folded string with spin in a plane

Rigid circular string rotating in two planes of R^4

$$t = \kappa\tau, \quad \mathrm{x}_1 \equiv x_1 + ix_2 = a\, e^{i(\tau+\sigma)}, \quad \mathrm{x}_2 \equiv x_3 + ix_4 = a\, e^{i(\tau-\sigma)}$$
$$E_{\rm flat} = \frac{\kappa}{\alpha'} = \sqrt{\frac{2}{\alpha'}N}\,, \qquad N = J_1 + J_2, \qquad J_1 = J_2 = \frac{a^2}{\alpha'}$$

semiclassical counterpart of quantum string state created by

$$e^{-iEt}\left[(\partial\mathrm{x}_1\bar\partial\mathrm{x}_1)^{\frac{J_1}{2}}(\partial\mathrm{x}_2\bar\partial\mathrm{x}_2)^{\frac{J_2}{2}} + ...\right]$$

Folded string rotating in a plane

$$t = \kappa\tau\,, \qquad \mathrm{x}_1 \equiv x_1 + ix_2 = a\sin\sigma\, e^{i\tau}$$
$$E_{\rm flat} = \sqrt{\frac{2}{\alpha'}N}\,, \qquad N = S = \frac{a^2}{2\alpha'}\,,$$

semiclassical counterpart of quantum string state

$$e^{-iEt}\left[(\partial\mathrm{x}_x\bar\partial\mathrm{x}_x)^{S/2} + ...\right]$$

3 ways to embed circular solutions into $AdS_5 \times S^5$:

(i) the two 2-planes in S^5: $J_1 = J_2$ "small string"
(ii) the two 2-planes in AdS_5: $S_1 = S_2$ "small string"
(iii) one plane in AdS_5 and the other in S^5: $S = J'$ "small string"

2 choices – AdS_5 or S^5 – for folded string

for $N = 2$ all 5 cases represent states on 1st string level;
for $N = J = 2$ they are particular members of Konishi multiplet
can be used to check universality of λ-dependent part of $\Delta = E$
for different states in supermultiplet

Spins: S_1, S_2 in AdS_5; (J_1, J_2) in S^5
orbital momentum $J = J_3$ in S^5

Examples:

- folded string with spin S_1 and momentum J:

$S_1 = J = 2 \rightarrow \quad [0,2,0]_{(1,1)}, \quad \Delta_0 = 4$

- folded string with spin J_1 and momentum J:

$J_1 = J = 2 \rightarrow \quad [2,0,2]_{(0,0)}, \quad \Delta_0 = 4$

- circular string with spins $J_1 = J_2$ and momentum J:

$J_1 = J_2 = 1, J = 2 \rightarrow \quad [0,1,2]_{(0,0)}, \quad \Delta_0 = 6$

- circular string with spins $S_1 = S_2$ and momentum J:

$S_1 = S_2 = 1, J = 2 \rightarrow \quad [0,1,2]_{(0,0)}, \quad \Delta_0 = 6$

- circular string with spins $S_1 = J_1$ and momentum J:

$S_1 = J_1 = 1, J = 2 \rightarrow \quad [1,1,1]_{(\frac{1}{2},\frac{1}{2})}, \quad \Delta_0 = 6$

Results: for several states on leading Regge trajectory

$$E^2 = 2\sqrt{\lambda}N + J^2 + n_{02}N^2 + n_{11}N + \frac{1}{\sqrt{\lambda}}\big(n_{01}J^2N + n_{03}N^3 + n_{12}N^2 + n_{21}N\big) + \frac{1}{(\sqrt{\lambda})^2}\big(\widetilde{n}_{11}J^2N + \widetilde{n}_{02}J^2N^2 + n_{04}N^4 + n_{13}N^3 + n_{22}N^2 + n_{31}N\big) + \frac{1}{(\sqrt{\lambda})^3}\big(\widetilde{n}_{01}J^4N + \widetilde{n}_{21}J^2N + \widetilde{n}_{12}J^2N^2 + n_{05}N^5 + ...\big) + O(\frac{1}{(\sqrt{\lambda})^4})$$

- $n_{01} = 1,\ \widetilde{n}_{01} = -\frac{1}{4}, ...$ from near-BMN expansion ($J \ll \sqrt{\lambda}$)

$$E^2 = J^2 + 2N\sqrt{\lambda + J^2} + ... = J^2 + N(2\sqrt{\lambda} + \frac{J^2}{\sqrt{\lambda}} + ...)$$

- "tree-level" coeffs $n_{02}, n_{03}, n_{04}, ...$ are all rational
- leading 1-loop n_{11} is rational [Roiban, AT 09; Gromov et al 11]
- $\widetilde{n}_{11} = -n_{11}$, i.e in general [BGMRT 12]

$$h_1 = 2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2} + \frac{n_{11}}{1+\mathcal{J}^2} + \frac{1}{\sqrt{\lambda}}\big[n_{21} + \widetilde{n}_{21}\mathcal{J}^2 + O(\mathcal{J}^4)\big] + O(\tfrac{1}{(\sqrt{\lambda})^2})$$

$$h_2 = \frac{n_{02} + \mathcal{J}^2}{1+\mathcal{J}^2} + \frac{1}{\sqrt{\lambda}}\big[n_{12} + \widetilde{n}_{12}\mathcal{J}^2 + O(\mathcal{J}^4)\big] + O(\tfrac{1}{(\sqrt{\lambda})^2})$$

• $n_{12} = n'_{12} - 3\zeta_3, \qquad n'_{12} = -\frac{3}{8} - 2n_{03}$ is rational

[Tirziu, AT 08; Roiban, AT 09; Gromov-Valatka 11]

ζ_3 term is universal for states on leading Regge trajectory

• $\widetilde{n}_{12} = \widetilde{n}'_{12} + 3\zeta_3 + \frac{15}{4}\zeta_5, \ \ \widetilde{n}'_{12}$ rational

• n_{1k} contains universal ζ_{2k-1} (universal UV $n \gg 1$ asymptotics)

e.g. $n_{13} = \widetilde{n}'_{12} + \widetilde{n}''_{1k}\zeta_3 + \frac{15}{4}\zeta_5$

• leading 2-loop coefficient n_{21} is universal: $n_{21} = -\frac{1}{4}$

for folded string state [Basso]; evidence from universality [BGMRT]

of the Konishi state energy ($J = N = 2$)

$$E_{N=J=2} = \sqrt[4]{\lambda}\Big[2 + \frac{b_1}{\sqrt{\lambda}} + \frac{b_2}{(\sqrt{\lambda})^2} + \frac{b_3}{(\sqrt{\lambda})^3} + ...\Big]$$

$$b_1 = 1 + n_{02} + \frac{1}{2}n_{11} = 2$$

$$b_2 = -\frac{1}{4}b_1^2 + 2n_{01} + 2n_{03} + n_{12} + \frac{1}{2}n_{21} = \frac{1}{2} - 3\zeta_3$$

$$b_3 = a_1 + a_2\zeta_3 + \frac{15}{2}\zeta_5 \ , \qquad ...$$

b_1, b_2: match TBA predictions interpolated to $\lambda \gg 1$

• need 2-loop string sigma model computation
to confirm universality of n_{21}, fix n_{22} → determine b_3

Some details:

• Circular rotating string in S^5 with $J_1 = J_2 \equiv J'$:
flat space $R_t \times R^4$: circular string solution

$$x_1 + ix_2 = \ a \, e^{i(\tau+\sigma)} \ , \qquad x_3 + ix_4 = \ a \, e^{i(\tau-\sigma)}$$

$$E = \sqrt{\tfrac{4}{\alpha'} J'}, \qquad J' = \tfrac{a^2}{\alpha'}$$

directly embedded into $R_t \times S^5$ in $AdS_5 \times S^5$ [Frolov, AT 03] :
string on small sphere inside S^5: $X_1^2 + ... + X_6^2 = 1$

$$X_1 + iX_2 = a \, e^{i(\tau+\sigma)}, \quad X_3 + iX_4 = a \, e^{i(\tau-\sigma)} \ ,$$
$$X_5 + iX_6 = \sqrt{1-2a^2}, \qquad t = \kappa\tau$$
$$\mathcal{J}' = \mathcal{J}_1 = \mathcal{J}_2 = a^2, \qquad \mathcal{E}_0^2 = \kappa^2 = 4\mathcal{J}'$$

E_0 is just as in flat space

$$E_0 = \sqrt{\lambda}\mathcal{E} = \sqrt{4\sqrt{\lambda}J'}\,, \qquad J' = \sqrt{\lambda}\mathcal{J}'$$

1-loop correction to energy: closed string ($R \times S^1$)

sum of bosonic and fermionic fluctuation frequencies ($n = 0, 1, 2, ...$)

Bosons (2 massless + massive):

$$AdS_5: \quad 4\times \qquad \omega_n^2 = n^2 + 4\mathcal{J}'$$
$$S^5: \quad 2\times \qquad \omega_{n\pm}^2 = n^2 + 4(1-\mathcal{J}') \pm 2\sqrt{4(1-\mathcal{J}')n^2 + 4\mathcal{J}'^2}$$

Fermions:

$$4\times \qquad \omega_{n\pm}^{2\,f} = n^2 + 1 + \mathcal{J}' \pm \sqrt{4(1-\mathcal{J}')n^2 + 4\mathcal{J}'}$$

$$E_1 = \frac{1}{2\kappa}\sum_{n=-\infty}^{\infty}\Big[4\omega_n + 2(\omega_{n+} + \omega_{n-}) - 4(\omega^f_{n+} + \omega^f_{n-})\Big]$$

expand in small $\mathcal{J}'$ and do sums (ζ_k come from $\sum_n$)

$$E_1 = \frac{1}{\sqrt{\mathcal{J}'}}\Big[\mathcal{J}' - (3+\zeta_3)\mathcal{J}'^2 - \tfrac{1}{4}\big(5 + 6\zeta_3 + 30\zeta_5\big)\mathcal{J}'^3 + \ldots\Big]$$

$$E = E_0 + E_1 = 2\sqrt{\sqrt{\lambda} J'} \Big[1 + \frac{1}{2\sqrt{\lambda}} - \frac{3}{4}(1+2\zeta_3)\frac{J'}{(\sqrt{\lambda})^2} + ... \Big]$$

To get a state on the first excited string level ($N = 2J'$)
should choose $J' = 1$, i.e. $J_1 = J_2 = 1$
for minimal non-trivial value of $J = J_3 = 2$
there is unique corresponding state in Konishi multiplet table:
$[0,1,2]_{(0,0)}$ at level $\Delta_0 = 6$ and thus

$$b_1 = 2\Big(\frac{J^2}{8J'} + \frac{1}{2}\Big)_{J=2, J'=1} = 2$$

- Small circular spinning string with $S_1 = S_2$

rigid circular string with two equal spins in AdS^5
and orbital momentum $J = J_3$ in S^5

$$Y_0 + iY_5 = \sqrt{1+2r^2}\, e^{i\kappa t} \ , \ \ Y_1 + iY_2 = r\ e^{i(w\tau+\sigma)} \ , \ \ Y_3 + iY_4 = r\ e^{i(w\tau-\sigma)}$$
$$X_5 + iX_6 = e^{i\nu\tau} \ , \qquad w^2 = \kappa^2 + 1 \ , \qquad \kappa^2(1+2r^2) = 2r^2(1+w^2) + \nu^2$$

$$\mathcal{E}_0 = (1+2r^2)\kappa = \kappa + \frac{2\kappa \mathcal{S}}{\sqrt{1+\kappa^2}} \,, \qquad \mathcal{S} = \mathcal{S}_1 = \mathcal{S}_2 = r^2 w \,, \qquad \mathcal{J} = \nu$$

"short" string expansion of the classical energy

$$E_0 = \sqrt{\lambda}\mathcal{E}_0 \,, \qquad \mathcal{E}_0 = 2\sqrt{\mathcal{S}} \Big(1 + \mathcal{S} + \frac{\mathcal{J}^2}{8\mathcal{S}} + ...\Big)$$

including 1-loop correction:

$$E_0 + E_1 = 2\sqrt{\sqrt{\lambda}S} \Big[1 + \frac{1}{\sqrt{\lambda}}\Big(S + \frac{J^2}{8S} - \frac{1}{2}\Big) + \mathcal{O}(\frac{1}{(\sqrt{\lambda})^2})\Big]$$

state on the first excited level $(N = 2S)$

has two excited oscillators, i.e. should have $S = S_1 = S_2 = 1$

for $J = 2$ the dual state in representation $[0,2,0]_{(1,0)}$

there is just one state in Konishi table with $\Delta_0 = 6$

$$b_1 = 2\Big(S + \frac{J^2}{8S} - \frac{1}{2}\Big)_{S=1,J=2} = 2$$

• Small circular spinning string with $S = J_1$ and $J = J_2 \neq 0$

rigid circular solution with one spin in AdS_5 and one spin in S^5

and orbital momentum J in S^5

$$Y_0 + iY_5 = \sqrt{1+r^2}\, e^{i\kappa t} \,, \qquad Y_1 + iY_2 = r\, e^{i(w\tau+\sigma)} \,, \qquad w^2 = \kappa^2 + 1 \,,$$
$$X_1 + iX_2 = a\, e^{i(w'\tau-\sigma)} \,, \qquad X_3 + iX_4 = \sqrt{1-a^2} e^{i\nu\tau} \,, \qquad w'^2 = \nu^2 + 1 \,,$$
$$\mathcal{E}_0 = 2\sqrt{\mathcal{S}} \left(1 + \frac{1}{2}\mathcal{S} + \frac{\mathcal{J}^2}{8\mathcal{S}} + ...\right)$$

The leading 1-loop correction to the energy vanishes

(cancellation of AdS and sphere contributions)

$$E_0 + E_1 = 2\sqrt{\sqrt{\lambda} S} \left[1 + \frac{1}{\sqrt{\lambda}}\Big(\frac{1}{2}S + \frac{J_2^2}{8S}\Big) + \mathcal{O}\big(\frac{1}{(\sqrt{\lambda})^2}\big)\right]$$

state on the first excited level: $S = J_1 = 1$

for $J = 2$ get state $[1,1,1]_{(\frac{1}{2},\frac{1}{2})}$ at $\Delta_0 = 6$ level

$$b_1 = 2\Big(\frac{1}{2}S + \frac{J^2}{8S}\Big)_{S=1, J=2} = 2$$

Conclusions

• beginning of understanding of $AdS_5 \times S^5$ string spectrum
= spectrum of conformal $\mathcal{N} = 4$ SYM operators

• agreement with numerical results from TBA:
non-trivial check of quantum integrability

• prediction of transcendental structure of leading coefficients:
reproduce them by an analytic solution of TBA at strong coupling?

• evidence of universality of some coefficients in strong coupling expansion of dimensions of states on leading Regge trajectory

• need systematic study of quantum string theory in $AdS_5 \times S^5$ in near-flat-space expansion

and of course we still need first-principles solution for the spectrum of $AdS_5 \times S^5$ superstring = spectrum of $\mathcal{N} = 4$ SYM based on integrability

it is now appearing to be within reach ....