# BPS States in N=4

**Ashoke Sen**

Harish-Chandra Research Institute, Allahabad, India

**Caltech, March 2012**

## Goal:

**Study the spectrum of BPS states in N=4 supersymmetric SU(n) Yang-Mills theories on the Coulomb branch where the gauge group is broken to $U(1)^{n-1}$.**

**1. What are the possible questions whose answers are protected from changes under continuous deformation of the moduli and coupling constants?**

**2. What are the answers to these questions?**

**Review + arXiv:1203.4889**

**We shall not follow the historical path but review the results from modern perspective.**

## The role of index

We would like to compute properties of BPS states which are protected from changes under continuous deformation of moduli and coupling constants.

– need appropriate supersymmetric index.

The appropriate index in D=4 is the helicity trace index.

Bachas, Kiritsis

Suppose we have a BPS state that breaks 4n supersymmetries.

$\rightarrow$ there will be 4n fermion zero modes (goldstino) on the world-line of the state.

Consider a pair of fermion zero modes $\psi_0$, $\psi_0^\dagger$ carrying $J_3 = \pm 1/2$ and satisfying

$$\{\psi_0, \psi_0^\dagger\} = 1$$

If $|0\rangle$ is the state annihilated by $\psi_0$ then

$$|0\rangle, \qquad \psi_0^\dagger |0\rangle$$

give a degenerate pair of states with $J_3 = \pm 1/4$ and hence

$$(-1)^F = (-1)^{2J_3} = (-1)^{\pm 1/2} = \pm i$$

Thus

$$\mathrm{Tr}(-1)^F = 0, \qquad \mathrm{Tr}(-1)^F (2J_3) = i$$

**Lesson: Quantization of the fermion zero modes produces Bose-Fermi degenerate states and makes $\mathrm{Tr}(-1)^F$ vanish.**

**Remedy: Define**

$$B_{2n} = \frac{(-1)^n}{(2n)!}\mathrm{Tr}(-1)^F(2J_3)^{2n} = \frac{(-1)^n}{(2n)!}\mathrm{Tr}(-1)^{2J_3}(2J_3)^{2n}$$

**Since there are 2n pairs of zero modes,**

$$\begin{aligned} B_{2n} &= \frac{(-1)^n}{(2n)!}\mathrm{Tr}_{rest}\mathrm{Tr}_{zero}(-1)^{2J_3^{(1)}+\cdots 2J_3^{(2n)}+2J_3^{rest}} \\ &\quad \times \left(2J_3^{(1)} + \cdots 2J_3^{(2n)} + 2J_3^{rest}\right)^{2n} \\ &= (-1)^n \mathrm{Tr}_{rest}\mathrm{Tr}_{zero}(-1)^{2J_3^{(1)}+\cdots 2J_3^{(2n)}+2J_3^{rest}} \times 2J_3^{(1)} \times \cdots 2J_3^{(2n)} \\ &= (-1)^n (i)^{2n} \times \mathrm{Tr}_{rest}(-1)^{2J_3^{rest}} \end{aligned}$$

$$B_{2n} = Tr_{rest}(-1)^{2J_3^{rest}}$$

Thus $B_{2n}$ effectively counts $Tr_{rest}(-1)^F$, with the trace taken over modes other than the 4n fermion zero modes associated with broken supersymmetry.

Note: $B_{2n}$ does not receive any contribution from non-BPS states which break more than 4n supersymmetries and hence have more than 4n fermion zero modes.

Due to this property $B_{2n}$ is protected from quantum corrections.

## Examples

N=4 SYM in D=4 has 16 supersymmetries.

1/2 BPS states break 8 supersymmetries.

$\Rightarrow$ the relevant index is $B_4$.

1/4 BPS states break 12 supersymmetries.

Thus the relevant index is $B_6$.

## Twisted index

Suppose the theory has a global symmetry g that commutes with some of the unbroken supersymmetries of the BPS state.

Suppose further that there are 4n broken g-invariant supersymmetries.

In that case

$$B^{g}_{2n} = \frac{(-1)^{n}}{(2n)!} \mathrm{Tr}\left[(-1)^{2h}(2h)^{2n} g\right]$$

is a protected index that carries information about g quanum number.

Note: Such global symmetries may appear in subspaces of the full moduli space in which case $B^{g}_{2n}$ can be defined only in that subspace.

We shall mostly focus on SU(n) SYM theories

– can be geometrically realized on the world-volume of n D3-branes.

Leaving out the 3+1 world-volume directions each D3-brane can be located at any point in the 6 transverse directions.

– 6n dimensional moduli space.

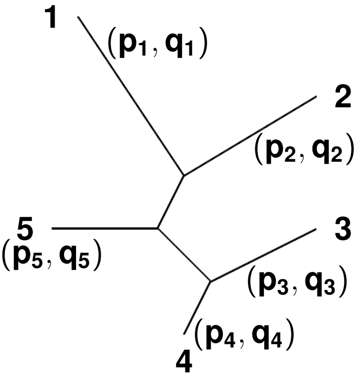
Of these 6 are associated with the center of mass U(1) SYM and are irrelevant for our problem.

Complex coupling constant:

$$\tau = \frac{\theta}{2\pi} + i\,\frac{4\pi}{g_{YM}^2}$$

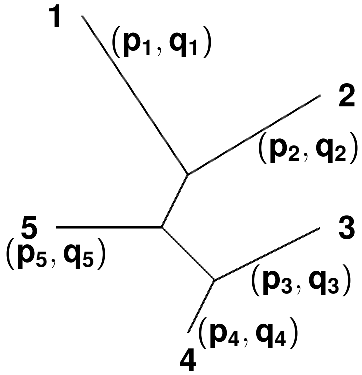
**In this description, BPS states are conveniently represented as string networks ending on D3-branes.**

Bergman; Bergman, Kol




**(1,0): D-string, (0,1): fundamental string**

**If gcd(p,q) = r then (p,q) $\equiv$ r copies of (p/r, q/r).**




– carries charges $Q = (q_1, \cdots q_5)$, $P = (p_1, \cdots p_5)$.

Note: $\sum_i q_i = 0 = \sum_i p_i \Rightarrow$ no U(1) charge

## Rules for string network

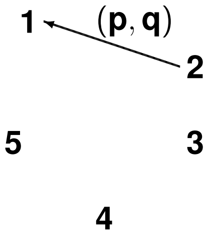
1. The network must be planar.

2. A (p,q) string should lie along $e^{i\alpha}(p\bar{\tau} + q)$ (class A) or $e^{i\alpha}(p\tau + q)$ (class B).

$\alpha$: a constant for a given network.

Schwarz; Aharony, Hanany, Kol; Dasgupta, Mukhi; A.S.

**Half BPS states correspond to single $(p, q)$ strings stretched between two D3-branes.**




$$Q = (q, -q, 0, 0, 0), \quad P = (p, -p, 0, 0, 0)$$

**Note: Q and P are always parallel.**

**For these states $B_4 = 1$ if gcd(q,p)=1 and 0 otherwise.**

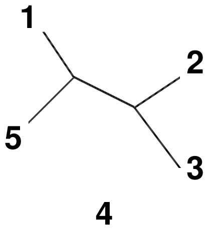
**– follows from S-duality invariance and known spectrum of (0,q) states.**

Olive, Witten; Osborn; A.S.; Segal, Selby

## Quarter BPS states

**The relevant index is $B_6$.**

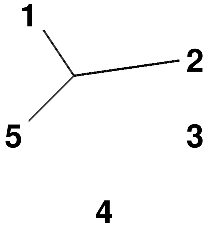
**A configuration with strings ending on four or more D-branes is non-planar at a generic point in the moduli space and hence non-supersymmetric.**




**$\Rightarrow$ has $B_6 = 0$.**

Thus $B_6$ receives contribution only from 3-string junctions.

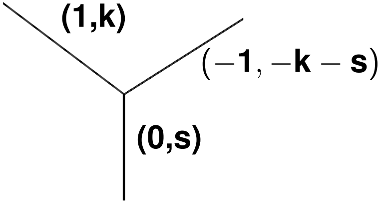
Bergman




We would like to compute $B_6$ for an arbitrary 3-string junction.

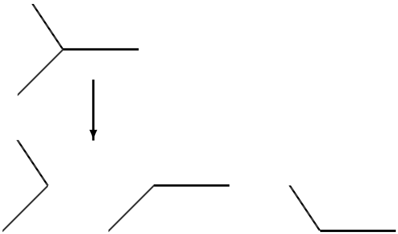
Although the string junction picture is useful in determining the region of existence of 1/4 BPS states, it is not very useful in finding the index $B_6$.

This can be done in special cases by representing them as bound states of multiple monopoles in gauge theories and solving the associated supersymmetric quantum mechanics problem. Bak, Lee, Yi; Gauntlett, Kim, Park, Yi; Stern, Yi




For this configuration $B_6 = (-1)^{s+1} s$

A three string junction exists in a certain region of the moduli space bounded by walls of margial stability.



Along each wall the 3-string junction becomes marginally unstable against decay into a pair of half BPS states.

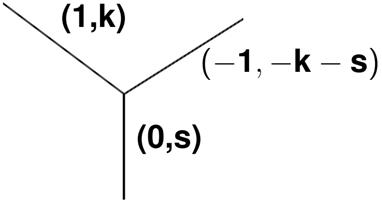
Beyond these walls the state stop existing.

This suggests that we can try to derive the index using wall crossing formula.

Across a wall along which (Q,P) decays into a pair of half BPS states with primitive charges $(Q_1, P_1)$ and $(Q_2, P_2)$ the jump in the index $B_6$ is

$$(-1)^{Q_1.P_2 - P_1.Q_2 + 1} |Q_1.P_2 - Q_2.P_1| B_4(Q_1, P_1) B_4(Q_2, P_2)$$

Denef; Denef, Moore; A.S.; Dabholkar, Gaiotto, Nampuri; Cheng, Verinde



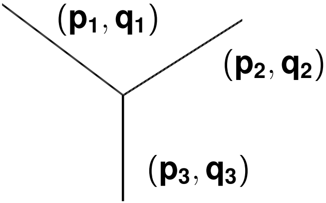

**Consider the wall along which the (0,s)≡s(0,1) string shrinks to zero-size.**

**– decays to (1,k) string connecting a pair of D3-branes and a $(-1, -k - s)$ string connecting a pair of D3-branes, each with $B_4 = 1$.**

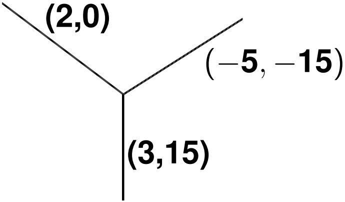
**The jump in $B_6$ across this wall is given by $(-1)^{s+1}s$ in agreement with direct index computation.**

**Dabholkar, Nampuri, Narayan**

Can we calculate the $B_6$ of a general 3-string junction?




In general the decays are not primitive, e.g. for




none of the decays are primitive.

We need to use general wall crossing formula for non-primitive decays in N=4 supersymmetric string / field theories.

Relevant decays are to pair of half BPS states:

$$(Q, P) \to (Q_1, P_1) + (Q_2, P_2)$$

$$\begin{aligned} \Delta B_6 &= (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1 + 1} |Q_1 \cdot P_2 - Q_2 \cdot P_1| \\ &\times \sum_{r_1|Q_1,P_1} B_4(Q_1/r_1, P_1/r_1) \sum_{r_2|Q_2,P_2} B_4(Q_2/r_2, P_2/r_2) \end{aligned}$$

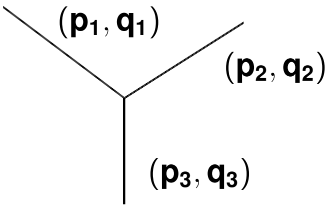
Banerjee, Srivastava, A.S; A.S.

For SYM theories,

$$\sum_{r|Q,P} B_4(Q/r, P/r) = 1$$

$\Rightarrow$ simple formula for $B_6$.

For




$$B_6 = (-1)^{p_1 q_2 - p_2 q_1 + 1} |p_1 q_2 - p_2 q_1|$$

– symmetric in the three strings.

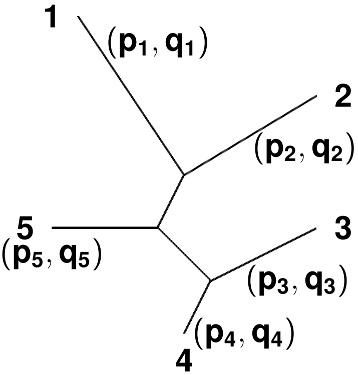
This solves completely the problem of computing $B_6$ for N=4 supersymmetric SU(n) theories.

This also agrees with the limits of the formula for $B_6$ in heteroric string theory on $T^6$.

Banerjee, Srivastava, A.S.

**For other gauge groups we can compute $B_6$ by identifying various SU(3) subgroups and computing $B_6$ for different chage vectors in each SU(3).**

We shall now return to more general planar string network (possibly with internal loops).




As argued before, $B_6$ and any other index that is defined everywhere in the moduli space vanish for this configuration.

## Strategy

**Introduce an index that can be defined only in spacial subspace of the moduli space where all the D3-branes lie in a plane.**

**⇔ only two of the six adjoint Higgs fields take vev.**

**Make use of the SO(4)$\equiv$ SU(2)$_L$ $\times$ SU(2)$_R$ rotational symmetry in the four directions transverse to the plane of the 3-branes to introduce a twisted index.**

## $SU(2)_L \times SU(2)_R \times SU(2)_{rotation}$ transformation laws of various supersymmetries:

| State | unbroken susy | broken susy |
|---|---|---|
| Half BPS | (1,2,2) + (2,1,2) | (1,2,2) + (2,1,2) |
| Class A quarter BPS | (1,2,2) | (1,2,2) + 2 (2,1,2) |
| Class B quarter BPS | (2,1,2) | 2(1,2,2) + (2,1,2) |

Class A states and half BPS states have four broken and four unbroken $SU(2)_L$ invariant SUSY

Of these two of each are invariant unde $I_{3R} + J_3$

$(I_{3L}, I_{3R}, J_3)$: Cartan gen. of $SU(2)_L \times SU(2)_R \times SU(2)_{rotation}$

Following earlier logic we can now introduce protected index

$$B_2(z) = -\frac{1}{2!}\mathrm{Tr}\{(-1)^{2J_3} z^{2I_{3L}} (2J_3)^2\}$$

$$B_1(y,z) = -\frac{1}{y-y^{-1}}\,\mathrm{Tr}\,\left\{e^{2i\pi J_3} z^{2I_{3L}}\, y^{2I_{3R}+2J_3}\,(2J_3)\right\}$$

Similar index for N=2 SUSY theories are known.

Gaiotto, Moore, Neitzke

One can show that for class A states

$$B_2(z) = \lim_{y\to 1} B_1(y,z)$$

$$B_6 = \lim_{z\to 1}(z+z^{-1}-2)^{-2} B_2(z)$$

Thus $B_1(y,z)$ is the most general index.

Can we compute $B_1(y,z)$ for a general string network?

1. For half BPS states we have all the information and $B_1(y,z)$ is straightforward to compute.

$$B_1(y,z) = (z + z^{-1} - y - y^{-1})(y - y^{-1})$$

for each primitive half BPS state.

2. For collinear configuration of D3-branes only half BPS states contribute to $B_1(y,z)$.

Strategy: Start from this collinear configuration and apply wall crossing formula to find the result elsewhere in the moduli space.

Caution: The structure of marginal stability walls is more complicated than before.

$B_1(y, z)$ follows a wall crossing formula similar to the motivic KS formula. Kontsevich, Soibelman

There are many physical ‘derivations’ of this formula by now.

Cecotti, Vafa; Gaiotto, Moore, Neitzke; Dimofte, Gukov; Andriyash, Denef, Jafferis, Moore; Lee, Yi; Kim, Park, Wang, Yi; Manschot, Pioline, A.S $\cdots$

We can try to derive the wall crossing formula for $B_1(y, z)$ using any of these approaches.

We use the last approach.

1. Given $\alpha = (\mathbf{Q}, \mathbf{P})$ and $\alpha' = (\mathbf{Q}', \mathbf{P}')$, define

$$\langle \alpha, \alpha' \rangle = \mathbf{Q}.\mathbf{P}' - \mathbf{Q}'.\mathbf{P}$$

2. Define

$$\bar{\mathbf{B}}_1(\alpha; \mathbf{y}, \mathbf{z}) \equiv \sum_{\mathbf{m}|\alpha} \mathbf{m}^{-1} \frac{\mathbf{y} - \mathbf{y}^{-1}}{\mathbf{y}^{\mathbf{m}} - \mathbf{y}^{-\mathbf{m}}} \mathbf{B}_1(\alpha/\mathbf{m}; \mathbf{y}^{\mathbf{m}}, \mathbf{z}^{\mathbf{m}})$$

3. Introduce the algebra:

$$[\mathbf{e}_\alpha, \mathbf{e}_{\alpha'}] = \frac{(-\mathbf{y})^{\langle \alpha, \alpha' \rangle} - (-\mathbf{y})^{-\langle \alpha, \alpha' \rangle}}{\mathbf{y} - \mathbf{y}^{-1}} \mathbf{e}_{\alpha+\alpha'}$$

Then wall crossing formula tells us that:

$$\mathbf{P}\left( \prod_{\alpha} \mathbf{exp}\left[ \bar{\mathbf{B}}_1(\alpha; \mathbf{y}, \mathbf{z}) \mathbf{e}_\alpha \right] \right)$$

is unchanged across a wall.

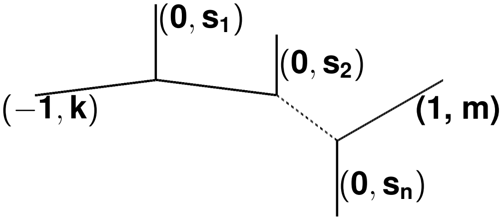
**P: ordering according to Arg(central charge).**

Using the wall crossing formula and the result for half BPS states we can compute $B_1(y,z)$ for any class A quarter BPS states in any chamber in the moduli space.

The corresponding index for class B states can be computed in a similar way.

## Example 1: Consider the network

Stern, Yi




**We can shrink the $(0, s_i)$ strings one by one and apply the primitive wall crossing formula.**

**Final result for $B_1(y, z)$:**

$$(-1)^{\sum |s_i|+n} \left\{ z + z^{-1} - y - y^{-1} \right\}^{n+1} \prod_i \frac{y^{|s_i|} - y^{-|s_i|}}{y - y^{-1}}$$

**We get the same result by shrinking the $(-1, k)$ string and applying semi-primitive wall crossing formula.**

$$B_1(y,z) = (-1)^{\sum |s_i| + n} \left\{ z + z^{-1} - y - y^{-1} \right\}^{n+1} \prod_i \frac{y^{|s_i|} - y^{-|s_i|}}{y - y^{-1}}$$

**Consistency check:**

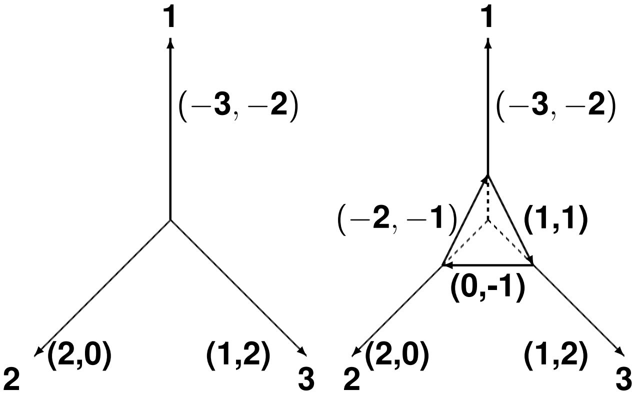
1. For $y = 1$, $z = -1$ we get $-4 \prod_i (-1)^{s_i} (4 s_i)$

– agrees with an appropriate index computed in supersymmetric quantum mechanics of multiple monopoles. Stern, Yi

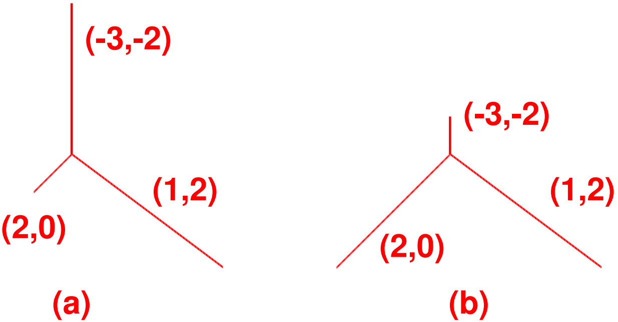
This has also been derived by applying primitive wall crossing formula. Dabholkar, Nampuri, Narayan

2. y-dependence agrees with quiver quantum mechanics analysis. Denef

**Example 2:**




**We can try to compute the index by shrinking any of the external strings to zero size.**

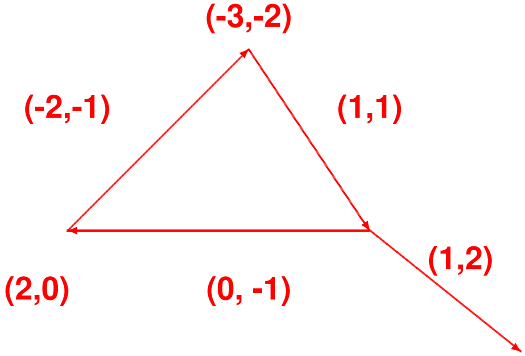



**Result for $B_1(y, z)$ for cases (a) and (b):**

$$\left\{z + z^{-1} - y - y^{-1}\right\}^2 \left\{-y^3 - \frac{1}{y^3} - y - \frac{1}{y}\right\}$$

$$\left\{z + z^{-1} - y - y^{-1}\right\}^2 \left\{-y^3 - \frac{1}{y^3} - 2y - \frac{2}{y} + z + \frac{1}{z}\right\}$$

**$\Rightarrow$ there must be marginal stability wall separating (a) and (b).**




– can break apart into the (-2,-1) string and the rest.

The jump in $B_1(y, z)$ across this wall precisely accounts for the previous difference.

## Conclusion

**The protected information about the spectrum of BPS states in $\mathcal{N} = 4$ supersymmetric Yang-Mills theories is encoded in the index $\mathbf{B_1(y, z)}$.**

**We now have a complete algorithm for computing this index for any charge vector at any point in the moduli space.**