Scale and Conformal Invariance in $d = 4$

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work with

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$\mathcal{N} = 4$ Super Yang-Mills Theory, 35 Years After

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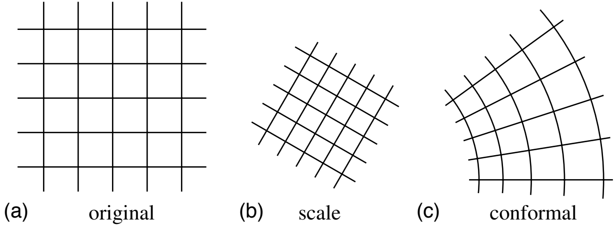
Overview: Scale vs. Conformal Invariance



It is difficult to find quantum field theories that are scale invariant without being fully conformal invariant.

In $d = 2$ there is a close connection between this question and the monotonicity of the RG flow (Zamolodchikov ’86; JP ’88)

We will try to exploit recent progress in the understanding of the RG flow in $d = 4$ (Komargodski & Schwimmer 1107.3987; Komargodski 1112.4538).

Overview, $\mathcal{N} = 4$

$d = 2$ Review

$d = 4$ Argument

Exceptions?

Scale vs. Conformal Invariance

A general scale current has the form (Wess)

$$S^\mu = T^\mu{}_\nu x^\nu + V^\mu \,,$$

where $T^\mu{}_\mu$ is the stress-energy tensor and V_μ is often called the virial current. Conservation of the scale current implies that

$$0 = \partial_\mu S^\mu \;\Rightarrow\; T^\mu{}_\mu = -\partial_\mu V^\mu$$

Scale invariance allows any vector operator V_μ, but in order to construct conformal generators we need the stronger condition

$$V_\mu = \partial_\nu L^{\mu\nu}$$

for some $L^{\mu\nu}$. (Equivalently, there is an improved $T_{\mu\nu}$ whose trace vanishes.)

Classical symmetries

Callan, Coleman and Jackiw ('70) and Coleman and Jackiw ('71) carried out this analysis at the classical level. They found that classically, any unitary renormalizable $d = 4$ field theory that is scale invariant is also conformally invariant.

They also noted nonunitary and nonrenormalizable exceptions, which will be discussed later.

But of course, scale and conformal transformations generically are quantum corrected, and the quantum situation is less straightforward.

Candidate virials

In many cases there simply is no nontrivial candidate for V^μ. For example, in Banks-Zaks, a non-Abelian gauge field coupled to N_f fundamentals and antifundamentals, with $N_f/N_c = \frac{11}{2} - \epsilon$: (1) The fermionic vector current j_V^μ is conserved. (2) The fermionic axial current j_V^μ is odd under CP; since we can rotate θ to zero the theory is CP-symmetric and the axial current cannot appear. (3) The Chern-Simons current $V_\mu dx^\mu = *\,\mathrm{Tr}(AdA + 2A^3/3)$ is not gauge invariant. So this theory is conformal.

But this argument is not general enough: by adding scalars and Yukawa couplings, we can get many cases where candidate virials exist but do not actually appear in S_μ.

$\mathcal{N} = 4$ SYM

By the way, how do we know that the birthday theory is actually conformally invariant? One possible virial is the gradient of the Konishi operator, $\partial_\mu(\Phi^a \Phi^a)$, but this can be improved away. But the fermionic $U(1)$ current

$$V_\mu = \bar{\psi}_i \sigma_\mu \psi_i\,?$$

is not conserved (broken by Yukawa couplings), so is a candidate. It is CP-odd, so can't appear at $\theta = 0$, but could conceivably be generated by instantons at nonzero θ.

But of course, supersymmetry will impose strong constraints...

Howe, Stelle & Townsend, Mandelstam '82 and Brink, Lindgren & Nilsson '82 prove perturbative finiteness — does this imply conformal invariance? Multiplet of anomalies (Sohnius and West '81) seems to show that $T^\mu{}_\mu = 0$, even nonperturbatively.

Modern argument (Seiberg '88): For scale invariance, go out on the Coulomb branch parameterized by Φ. Effective gauge Lagrangian is

$$\frac{1}{2g^2(\Phi)} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} \ .$$

$\mathcal{N}=4$ forbids a field-dependent kinetic term, so g^2 must be independent of scale. If there is a nontrivial virial, the effective action must reflect this through a term of the form

$$\frac{1}{\Phi} \partial_\mu \Phi V^\mu .$$

Again this is $\sim$ nonlinear two-derivative term, forbidden by $\mathcal{N}=4$.

Review of $d = 2$ argument

The arguments thus far are rather special. In $d = 2$ there is a general result. From conservation of $T_{\mu\nu}$,

$$x^\mu \frac{\partial}{\partial x^\mu} c(x) = -24 z^2 \bar{z}^2 G_{z\bar{z}, z\bar{z}} \,, \tag{1}$$

where $G_{\mu\nu,\sigma\rho} = \langle 0 | T_{\mu\nu}(x) T_{\sigma\rho}(0) | 0 \rangle$ and

$$c(x) = 2z^4 G_{zz,zz} - 4z^3 \bar{z} G_{z\bar{z},zz} - 6z^2 G_{z\bar{z},z\bar{z}} \,. \tag{2}$$

We are using the improved energy-momentum tensor, which scales canonically. In a scale invariant theory, $c(x)$ is independent of the separation, so $\langle 0 | T_{z\bar{z}}(x) T_{z\bar{z}}(0) | 0 \rangle = 0$, implying that in a unitary theory the trace $T_{z\bar{z}}$ vanishes.

In a nonunitary theory there might be exceptions, as the vanishing of the two-point function of the trace need not imply the vanishing of the trace: we will see an example later.

Exceptions can also arise if the field theory does not have a normalizable ground state, so that the two-point function does not exist. One class consists of nonlinear sigma models whose target space geometry satisfies

$$R_{ij} = \nabla_{(i}\xi_{j)}$$

(Hull and Townsend; Tseytlin '86) Conformal invariance requires further that ξ_i be a gradient $\partial_i f$. Such manifolds are necessarily noncompact, there is no normalizable ground state.

$d = 4$: Brief KS review

Komargodsky and Schwimmer monotonicity argument in $d = 4$: put a QFT in a background metric $\hat{g}_{\mu\nu}(x)$. Split

$$\hat{g}_{\mu\nu}(x) = e^{-2\tau(x)} g_{\mu\nu}(x)\,.$$

Expand around $g_{\mu\nu} = \eta_{\mu\nu}$ and look at the 4-dilaton S-matrix. On-shell condition is

$$\Box\varphi = 0\,, \quad \varphi/f = 1 - e^{-\tau}\,;$$

(dilaton = background field). If the QFT is conformal (over some range of scales), φ decouples from it and the $\varphi\varphi \to \varphi\varphi$ amplitude comes only from the effective local Wess-Zumino term $aS_{\mathrm{WZ}} =$

KS review

$$a S_{\mathrm{WZ}} = a \int d^4x \sqrt{g} \tau E_4 + \ldots$$
$$\to -2a \int d^4x \, (\partial_\mu \varphi \partial^\mu \varphi)^2 \, .$$

The anomaly σE_4 anomaly under

$$\tau \to \tau + \sigma \, , \quad g_{\mu\nu} e^{2\sigma}$$

is the same at all scales (zero, here) and equal to $a_{\mathrm{CFT}} - a$ at scales where the theory is conformal.

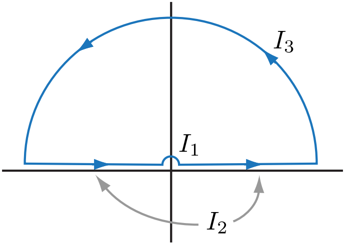
By a dispersive argument, a decreases toward the IR, so

$$a_{\mathrm{CFT,IR}} < a_{\mathrm{CFT,UV}} \, .$$

Evaluate

$$\int \frac{ds}{s^3} \mathcal{M}_{\varphi\varphi\to\varphi\varphi}$$

on



The integral on I_2 is nonnegative and (positive if the cross-section is nonzero).

Now, use these ingredients to address scale vs. conformal.

Scale vs. conformal

(i) In a conjectured SFT (scale but not conformal theory), the dilaton will not decouple (Nakayama, 1110.2586):

$$\tau T^{\mu}{}_{\mu} = -\tau \partial_{\mu} V^{\mu} \,.$$

The $\tau\tau \rightarrow$ SFT cross section is then generically nonzero, and by scale invariance $\propto f^{-4} s$. The dispersive argument then implies a logarithmic running of a.

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(ii) Such a logarithmic running is inconsistent, so an SFT cannot exist.

Inconsistency of running of a

Expand on (ii): consider the theory constructed in the background $\hat{g}_{\mu\nu}$ and defined by a UV cutoff (or flow from a UV CFT) at scale Λ.
At a much lower scale μ, the effective lagrangian will contain a term

$$C \log(\Lambda/\mu)(\partial_\mu\varphi\partial^\mu\varphi)^2 \, .$$

The UV-divergent part must be completed to an interaction that is (a) local and (b) a function only of $\hat{g}_{\mu\nu}$. But it can't: it comes only from a local term that has an anomalous variation.

Perturbative fixed points

(i) It remains to show that the log divergence is actually there. For weakly coupled CFT's we can do this explicitly. The leading coupling of the dilaton is through

$$\tau T^{\mu}{}_{\mu} = \left(\varphi + \tfrac{1}{2}\varphi^2 + \dots\right) T^{\mu}{}_{\mu}$$

Here

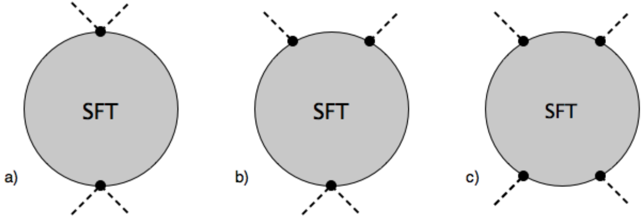
$$T^{\mu}{}_{\mu} = \sum_A \beta_A \mathcal{O}_A \overset{\text{SFT}}{=} g_A Q_{AB} \mathcal{O}_B \,.$$

Next order is of the form

$$\tau^2 (\partial_\mu^2 g_A) \mathcal{O}_A \sim \varphi^2 \beta_B \partial_{g_B} \beta_A \mathcal{O}_A$$

and is suppressed by $\partial_{g_B} \beta_A \ll 1$ at a weakly coupled fixed point.

Perturbative fixed points

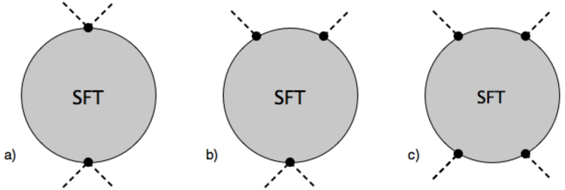


Each vertex brings in one factor of $T^{\mu}{}_{\mu} \propto \beta$, so (a) dominates at weak coupling,

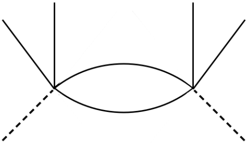
$$\int d^4x\, e^{ip\cdot x} \beta_A^2 \langle 0|\mathcal{O}_A(x)\mathcal{O}_A(0)|0\rangle = c_A \beta_A^2 \int d^4x \frac{e^{ip\cdot x}}{x^{-8}} \propto c_A \beta_A^2 s^2 \ln \frac{\Lambda}{s} \,.$$

(Komargodsky 1112.4538). QED.

Perturbative fixed points



We can make an independent argument based on (c), because (a) and (b) constribute only to s-wave scattering. For $l \geq 1$, (c) makes an unambiguously positive contribution to the cross section, e.g.



Perturbative fixed points

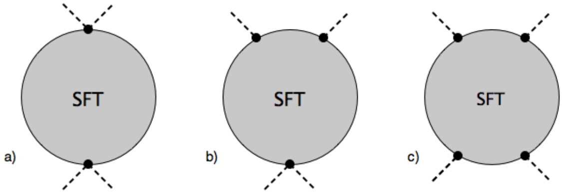
Either way, weakly coupled SFT's are ruled out.

Theories with discrete scale invariance also excluded, as is any flow for which

$$\int \frac{d\mu}{\mu} \beta^2$$

diverges in either the IR or the UV.

Strongly coupled fixed points



Could these cancel, for all intermediate states? It seems unlikely. It requires, e.g.

$$\langle 0|\mathrm{T}\, T^{\mu}{}_{\mu}(x_1)T^{\nu}{}_{\nu}(x_2)T^{\sigma}{}_{\sigma}(x_3)T^{\rho}{}_{\rho}(x_4)|0\rangle$$
$$= \frac{\text{const}}{(x_1 - x_2)^4 (x_3 - x_4)^4} + 2 \text{ perms.} + \text{just-right contact terms}\,,$$

Impossible?

Exceptions? I. Nonrenormalizable scalar theory

Coleman and Jackiw '71:

$$\mathcal{L} = \partial\Phi\partial\Phi + \frac{(\partial\Phi\partial\Phi)^2}{\Phi^4}\,.$$

Nonrenormalizable, but could be treated as an effective Lagrangian below the scale Φ. The background Φ breaks scale invariance, so

$$\sigma_{\varphi\varphi\to\mathrm{SFT}} \propto \frac{s^5}{f^4\Phi^8}$$

(rather than s/f^4). Octic divergence rather than a log, canceled by Φ^{-8}, leaving $O(1)$ which can be canceled by UV threshold contribution: no contradiction.

$\Phi \neq 0$: *there is no scale invariant background.*

Exceptions? II. Elasticity

Theory of elasticity (Callan, Coleman and Jackiw ’70; Riva and Cardy hep-th/0504197). Displacement $u_\mu(x)$ with action

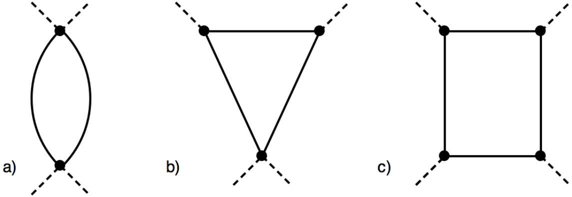
$$\int d^d x \sqrt{g} \left[\frac{1}{4} (\partial_\mu u_\nu - \partial_\nu u_\mu)(\partial^\mu u^\nu - \partial^\nu u^\mu) + \frac{h}{2} (\nabla_\mu u^\mu)^2 \right] .$$

Not reflection positive/unitary. Scale but not conformally invariant in any d. In $d = 4$ the couplings to the dilaton are

$$2h \left(\partial_\mu u^\mu \, u^\nu \partial_\nu \varphi + \partial_\mu u^\mu \, \varphi \, u^\nu \partial_\nu \varphi + (u^\mu \partial_\mu \varphi)^2 \right) .$$

Even though it is nonunitary, it provides a check on argument (ii): the log divergence should not appear.

Exceptions? II.



Individual graphs have log divergences proportional to $a + bh + ch^2$, but the sum is zero. Exclusive amplitudes are nonzero and have both local and contact pieces, but the total ‘cross section’ is zero.

Exceptions? III. $d = 4 - \epsilon$

Fortin, Grinstein, Stergiou (1106.2540, 1107.3840, 1202.4757) have found weakly coupled SFT's in $4 - \epsilon$ dimensions. This requires theories with scalars and fermions, and two-loop β-functions.

It is does not seem that our argument extends to $d = 4 - \epsilon$. The log divergence becomes $1/\epsilon$, but it multiplies β which is of order ϵ^2. Thus the amplitude could be completed by a nonlocal coupling to $g_{\mu\nu}$.

Exceptions? IV. $d = 4$ BZFGS

FGS also have at least one weakly coupled candidate example in $d = 4$, using a Banks-Zaks sector to provide a small parameter.

This example has a potential unbounded below, but this should not invalidate our argument based on UV divergences.

There is the possibility that three-loop terms will make V_μ cancel. Our argument predicts that this will occur.

Conclusions

The arguments, both for monotonicity and nonexistence of SFT's, are very dimension-dependent. Is there a general principle?

We have illuminated one small corner of the space of possible quantum field theories. Thanks in large part to lessons learned from $\mathcal{N} = 4$, quantum field theory still retains a primary role in the search for a 'theory of everything.'