Spectral Networks and Their Applications

Caltech, March, 2012

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Spectral Networks and Snakes, pretty much finished

Spectral Networks, almost finished

Wall-crossing in Coupled 2d-4d Systems: 1103.2598

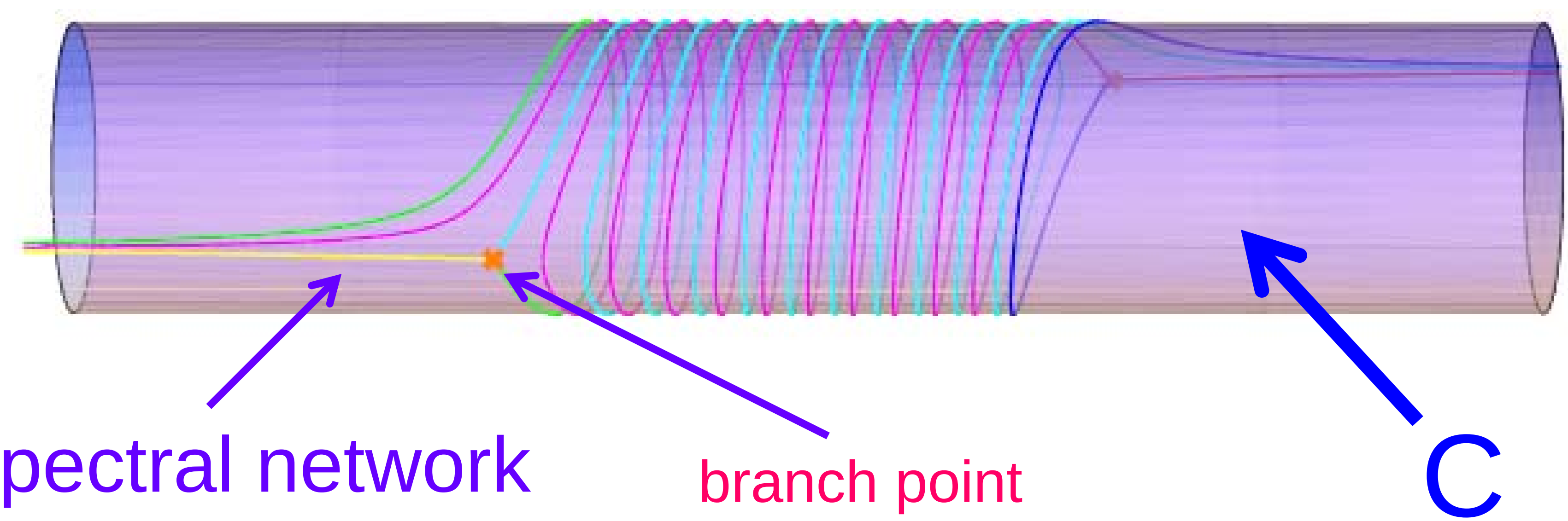
Framed BPS States: 1006.0146

Wall-crossing, Hitchin Systems, and the WKB Approximation: 0907.3987

Four-dimensional wall-crossing via three-dimensional field theory: 0807.4723

What are spectral networks?

Spectral networks are combinatorial objects associated to a covering of Riemann surfaces $\Sigma \longrightarrow C$



What are spectral networks good for?

They determine BPS degeneracies in D=4, N=2 field theories of class S. (this talk)

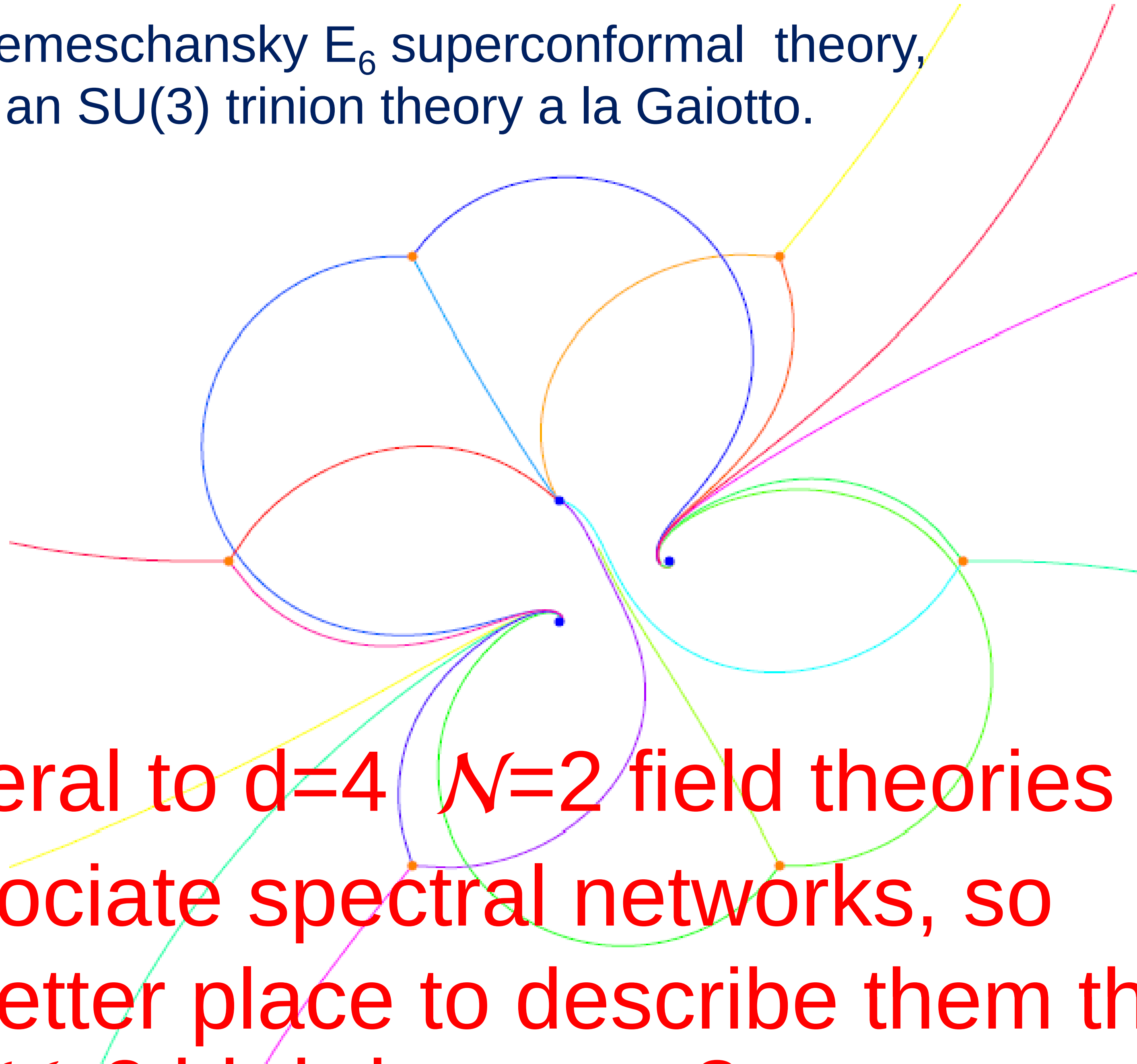
They give a "pushforward map" from flat U(1) gauge fields on Σ to flat nonabelian gauge fields on C.

They determine cluster coordinates on the moduli space of flat GL(K,$\mathbb{C}$) connections over C.

"Fock-Goncharov coordinates" "Higher Teichmuller theory"

Higher rank WKB theory

Minahan-Nemeschansky E_6 superconformal theory, realized as an SU(3) trinion theory a la Gaiotto.



In general to d=4 $\mathcal{N}$=2 field theories we can asociate spectral networks, so what better place to describe them than at the $\mathcal{N}$=2 birthday party?

Conference "*$\mathcal{N}$=4 Super Yang-Mills Theory, 35 Years After*"

OOPS….

But $\mathcal{N}$=4 SYM is ``really'' about the UV complete 6d theory S[$\mathfrak{g}$] with (2,0) susy….

To get $\mathcal{N}$=4 SYM we compactify S[$\mathfrak{g}$] on a torus….

A natural generalization is to ``theories of class S''

Outline

- Introduction
- Theories of class S & their BPS states
- Line defects and framed BPS states
- Surface defects & susy interfaces
- Spectral networks
- Determining the BPS degeneracies
- Conclusion

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Theories of Class S

Consider 6d nonabelian (2,0) theory S[$\mathfrak{g}$] for ``gauge algebra'' $\mathfrak{g}$

The theory has half-BPS codimension two defects D

Compactify on a Riemann surface C with D_a inserted at punctures z_a

$$so(5)_R \to so(3)_R \oplus \underbrace{so(2)_R}_{\text{Twist to preserve d=4,N=2}}$$

Witten, 1997
GMN, 2009
Gaiotto, 2009

$$S[\mathfrak{g}, C, D]$$

Type II duals via ``geometric engineering''
KLMVW 1996

Relation to Hitchin System

$$\begin{tikzcd}
 & S[\mathfrak{g}] \arrow[dl, "C"'] \arrow[dr, "S^1"] & \\
\boxed{S[\mathfrak{g}, C, D]} \arrow[dr, "S^1"'] & & \boxed{\text{5D } \mathfrak{g} \text{ SYM}} \arrow[dl, "C"] \\
 & \text{σ-Model:} \quad \mathbb{R}^{1,2} \to \mathcal{M} &
\end{tikzcd}$$

$$F + R^2[\varphi, \bar{\varphi}] = 0$$

$$\bar{\partial}_A \varphi = 0$$

Defects

$$\varphi \sim \frac{dz}{(z-z_a)^\ell}\mathfrak{r} + \cdots \qquad \ell \geq 1$$

Physics depends on choice of ℓ & $\mathfrak{r}$

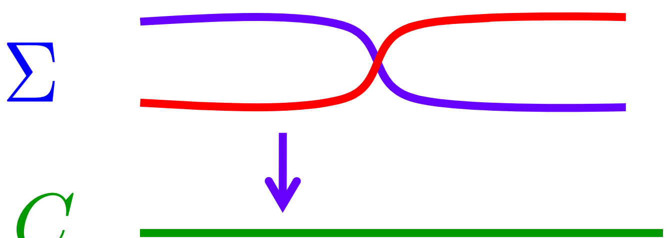
Seiberg-Witten Curve

UV Curve

$$\Sigma : \det(\lambda - \varphi(z,\bar{z})) = 0 \subset T^*C$$

$\lambda = p\,dq \qquad \lambda|_{\Sigma}$ SW differential

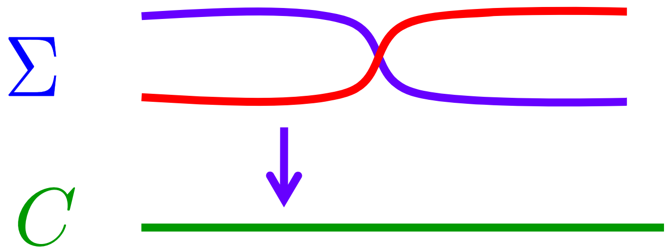
For $\mathfrak{g}$=su(K) $\qquad \pi : \Sigma \to C$



is a K-fold branched cover

$$\lambda^K + \lambda^{K-2}\phi_2(z) + \cdots + \phi_K(z) = 0$$

Coulomb Branch & Charge Lattice



IR theory is a (self-dual)
$\mathcal{N}$=2 abelian gauge theory

Coulomb branch

$$\mathcal{B} = \{u = (\phi_2, \dots, \phi_K)\}$$

Local system of charges

$$\Gamma = H_1(\Sigma; \mathbb{Z})$$

(Actually, Γ is a subquotient. Ignore that for this talk.)

BPS States: Geometrical Picture

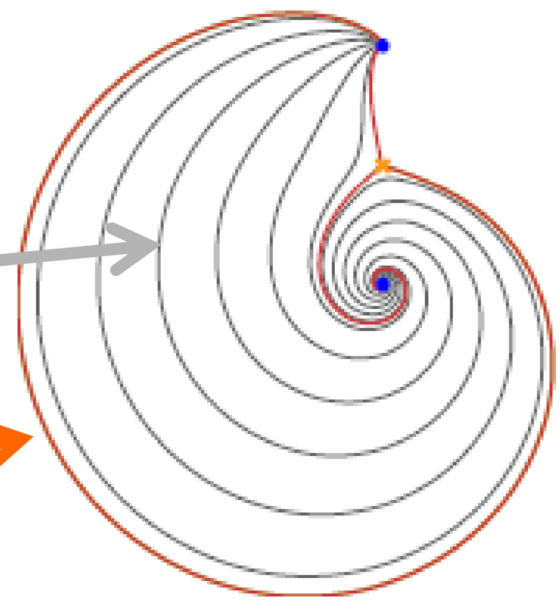
BPS states come from open M2 branes stretching between sheets i and j. Here i,j, =1,…, K. This leads to a nice geometrical picture with string webs:

Klemm, Lerche, Mayr, Vafa, Warner; Mikhailov; Mikhailov, Nekrasov, Sethi,

A ***WKB path*** of phase ϑ is an integral path on C

$$\langle \lambda_i - \lambda_j , \partial_t \rangle = e^{i\vartheta}$$

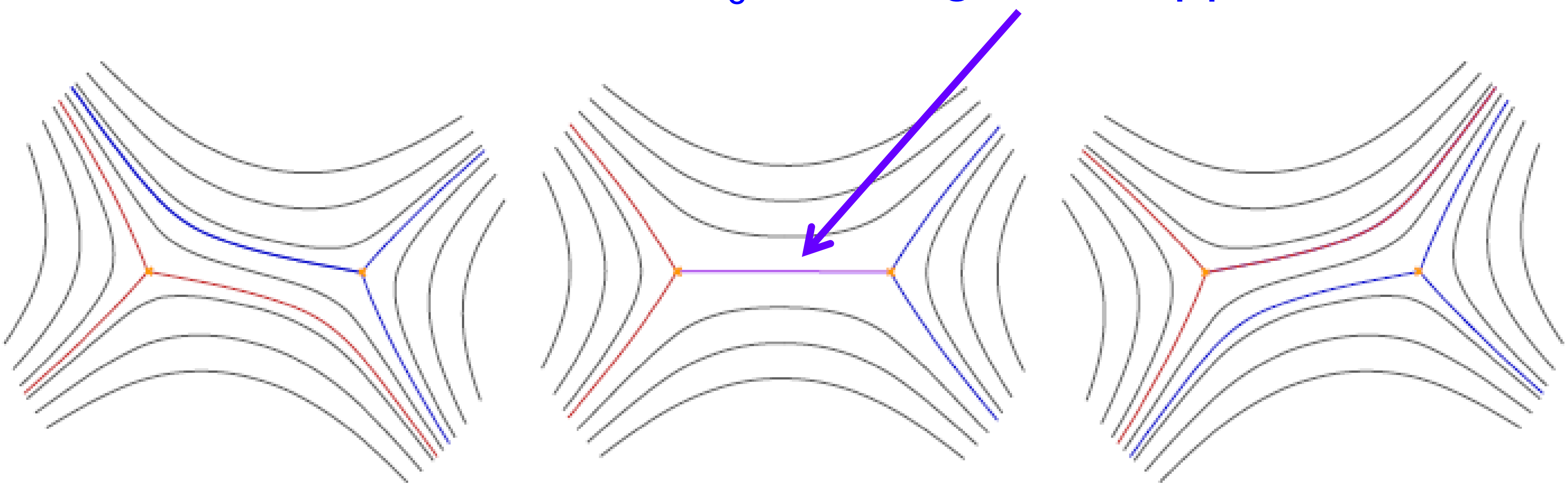
Generic WKB paths have both ends on singular points z_a



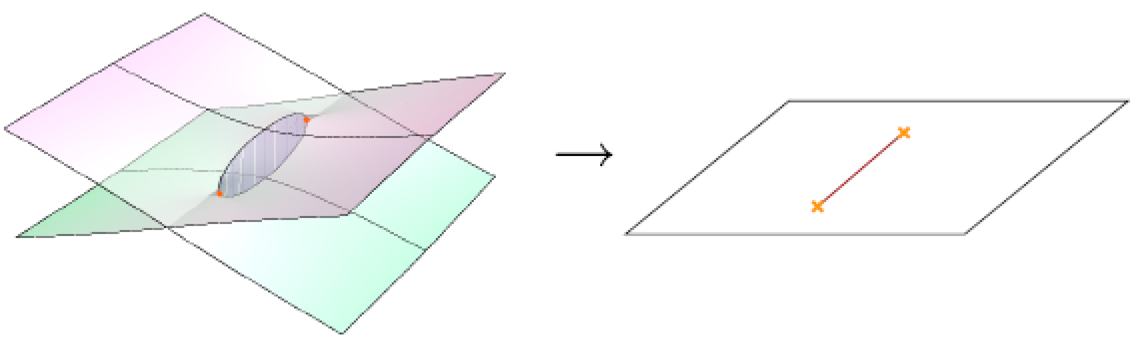
Separating WKB paths begin on branch points, and for generic ϑ, end on singular points

String Webs – 1/4

But at critical values of $\vartheta=\vartheta_c$ ``string webs appear'':

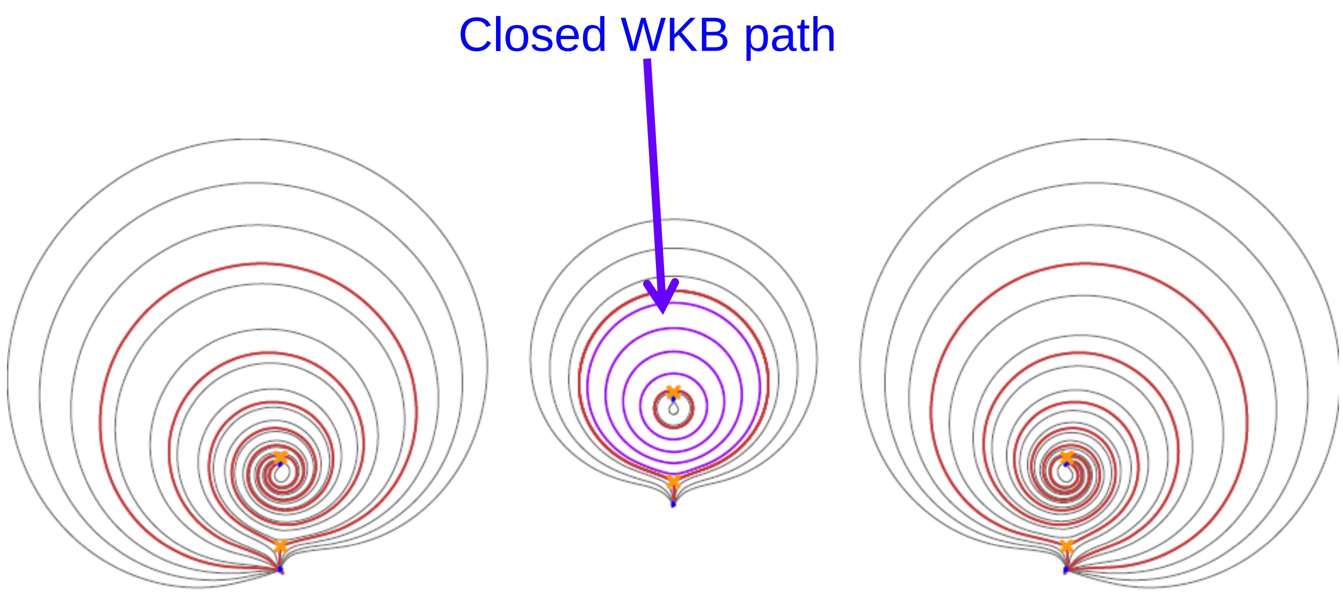


$\vartheta < \vartheta_c$ $\vartheta = \vartheta_c$ $\vartheta > \vartheta_c$

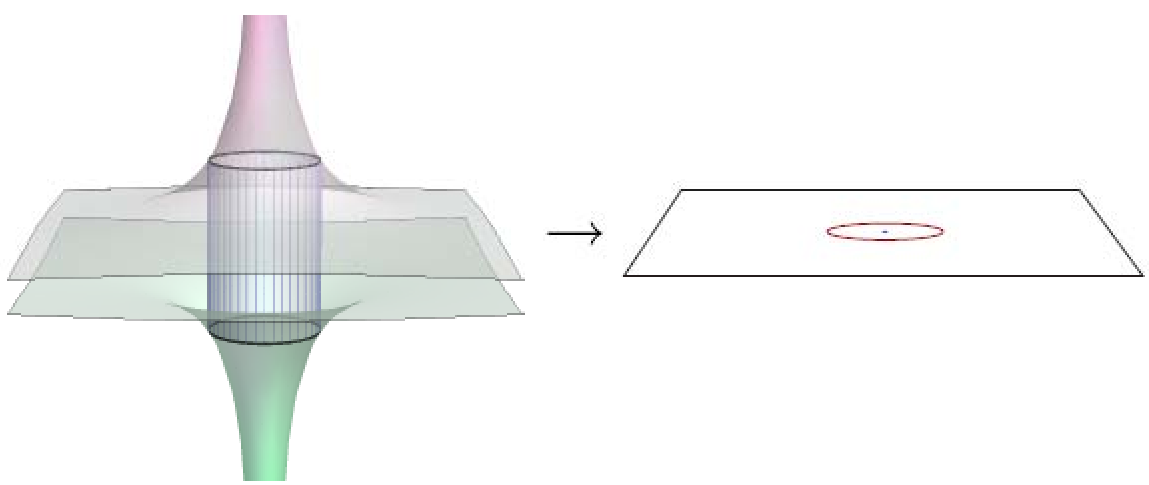


Hypermultiplet

String Webs – 2/4



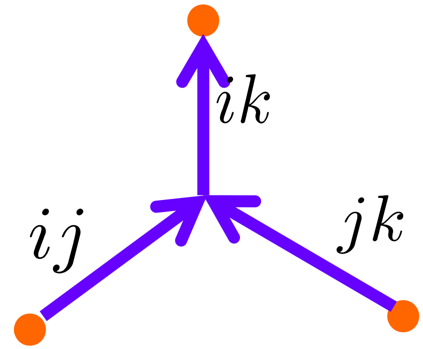
$\vartheta < \vartheta_c$ $\qquad$ $\vartheta = \vartheta_c$ $\qquad$ $\vartheta > \vartheta_c$



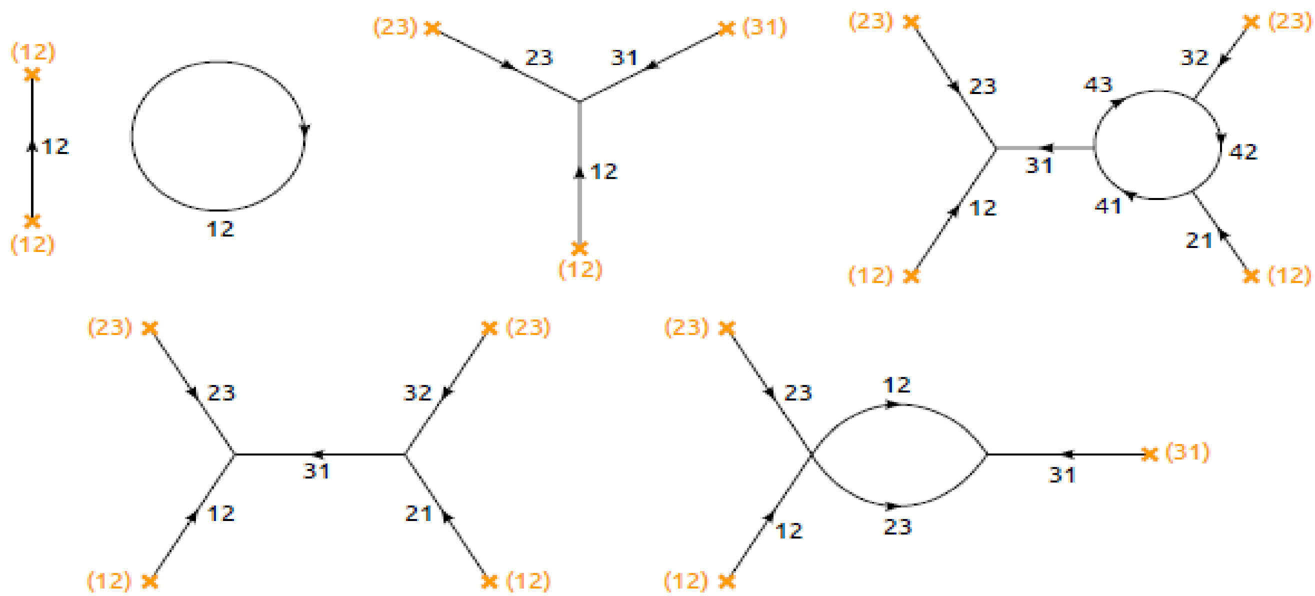
Vectormultiplet

String Webs – 3/4

At higher rank, we get string *junctions* at critical values of ϑ:

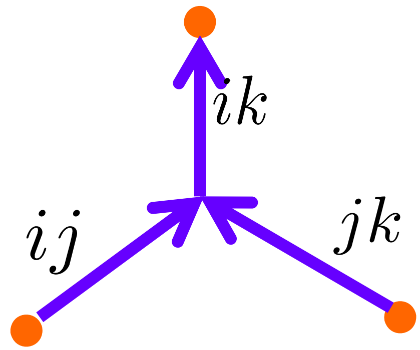


A ``***string web***'' is a union of WKB paths with endpoints on branchpoints or such junctions.



String Webs – 4/4

At higher rank, we get string *junctions* at critical values of ϑ:



A ``***string web***'' is a union of WKB paths with endpoints on branchpoints or such junctions.

These webs lift to closed cycles γ in Σ and represent BPS states with

$$Z_\gamma = \oint_\gamma \lambda = e^{i\vartheta_c}|Z_\gamma|$$

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Line Defects & Framed BPS States (in general)

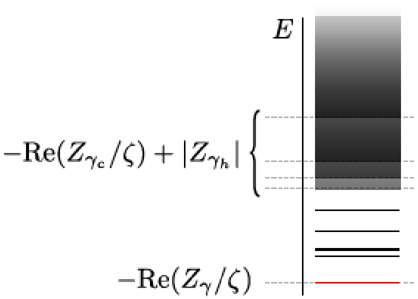
A ***line defect*** L (say along $\mathbb{R}_t$ x {0 }) is ***of type*** $\zeta = e^{i\vartheta}$ if it preserves the susys:

$$Q_\alpha^A + \zeta \sigma^0_{\alpha\dot{\beta}} \bar{Q}^{\dot{\beta}A}$$

Example:

$$L_\zeta = \exp \int_{\mathbb{R}_t \times \vec{0}} \left(\frac{\varphi}{2\zeta} + A + \frac{\zeta}{2}\overline{\varphi} \right)$$

$$\mathcal{H}_L = \oplus_{\gamma \in \Gamma} \mathcal{H}_{L,\gamma}$$



$$E \geq -\mathrm{Re}(Z_\gamma/\zeta)$$

Framed BPS States saturate this bound, and have framed protected spin character:

$$\overline{\underline{\Omega}} := \mathrm{Tr}_{\mathcal{H}^{bps}_{L,\gamma}} (-1)^{2J_3} (-y)^{2J_3+2I_3}$$

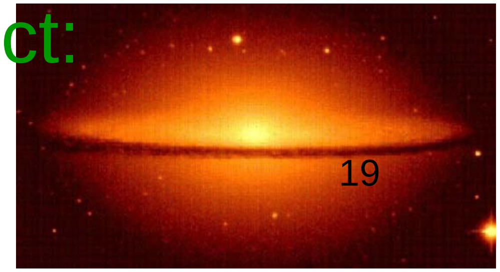
$$\overline{\underline{\Omega}}(L, \gamma; y; \zeta; u)$$

Piecewise constant in ζ and u, but has wall-crossing across ``BPS walls'' (for $\Omega(\gamma) \neq 0$):

$$W_\gamma := \{(u,\zeta) : Z_\gamma(u)/\zeta \in \mathbb{R}_-\}$$

Particle of charge γ binds to the line defect:

Similar to Denef's halo picture



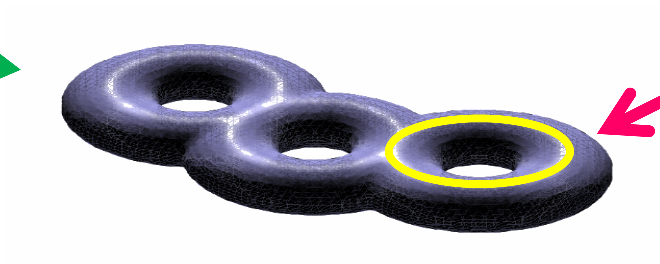
Line defects in S[$\mathfrak{g}$,C,D]

6D theory S[$\mathfrak{g}$] has supersymmetric surface defects S($\mathcal{R}$, σ)

For S[$\mathfrak{g}$,C,D] consider

$$\sigma = \mathbb{R} \times \{\vec{0}\} \times \wp$$

C



$$L_\zeta(\mathcal{R}, \wp)$$

Line defect in 4d *labeled* by isotopy class of a ***closed*** path $\wp$ and $\mathcal{R}$

k=2: Drukker, Morrison, Okuda

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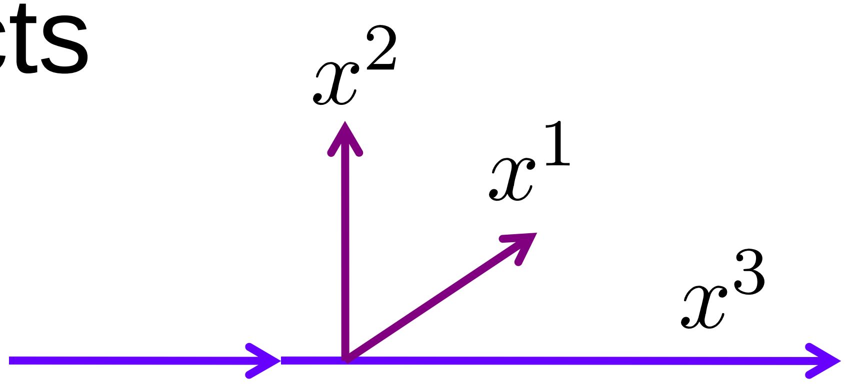
Surface defects (in general)



$\mathbb{S}$ at $x^1 = x^2 = 0$

UV Definition:

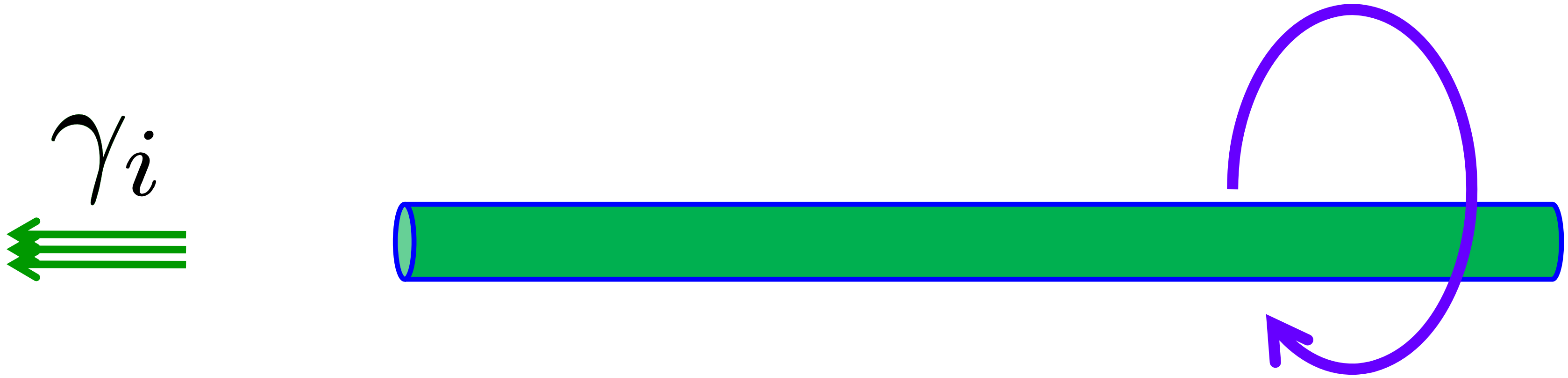
Preserves d=2 (2,2) supersymmetry subalgebra

Twisted chiral multiplet : $\Upsilon = \varphi + \cdots$

IR Description: ***Coupled 2d/4d system***

$$S_{IR} = \int d^4x d^4\theta \, \mathcal{F}^{eff}(a) + \int d^2x d^2\theta \, \mathcal{W}^{eff}(\Upsilon)$$

IR: Effective Solenoid

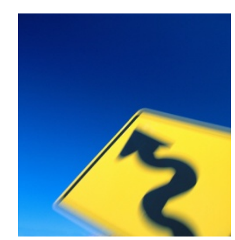


$$\oint \mathbb{A} = \gamma_i \in V = \Gamma_g \otimes \mathbb{R}$$

Introduce duality frame:

$$\gamma_i = \eta_I e^I + \alpha^I e_I$$

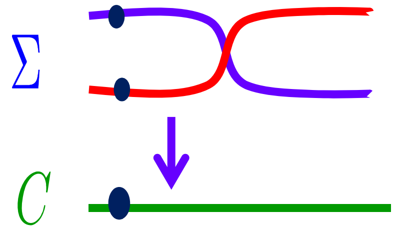
$$\eta + \tau \cdot \alpha = \frac{\partial \mathcal{W}^{eff}}{\partial a} = \frac{\partial Z_{\gamma_i}}{\partial a}$$



Canonical Surface Defect in S[$\mathfrak{g}$,C,D]

For z $\in$ C we have a ***canonical surface defect*** $\mathbb{S}_z$

It can be obtained from an M2-brane ending at $x^1=x^2=0$ in $\mathbb{R}^4$ and z in C



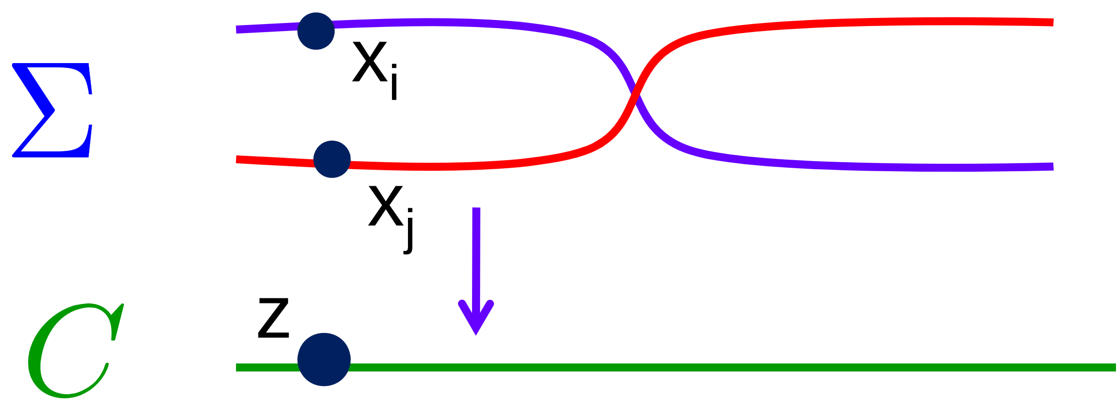
In the IR the different vacua for this M2-brane are the different sheets in the fiber of the SW curve over z.

Therefore the chiral ring of the 2d theory should be the same as the equation for the SW curve!

Alday, Gaiotto, Gukov, Tachikawa, Verlinde; Gaiotto

$$\lambda^K + \lambda^{K-2}\phi_2(z) + \cdots + \phi_K(z) = 0$$

Superpotential for $\mathbb{S}_z$ in S[$\mathfrak{g}$,C,D]

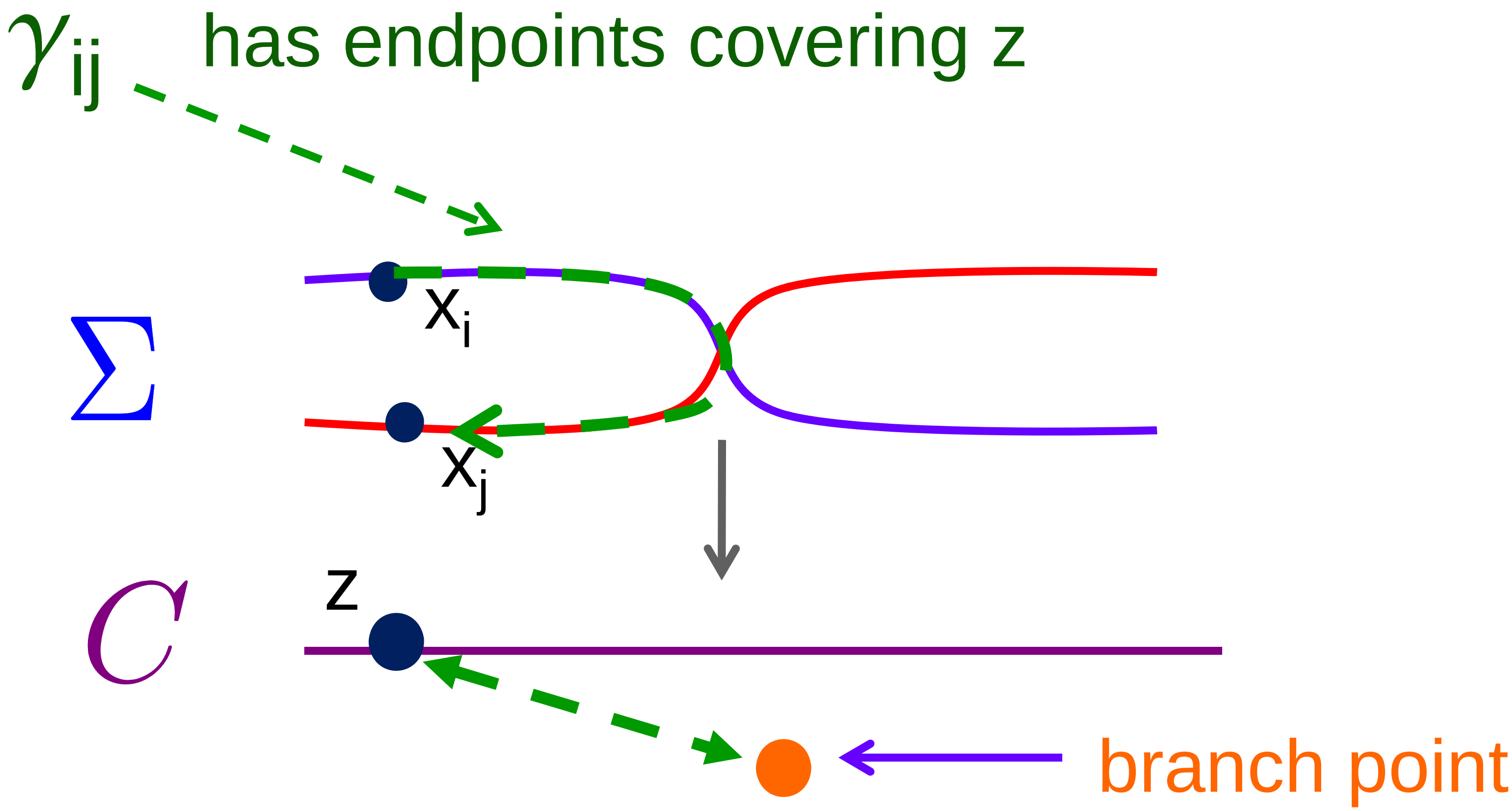


$$\boxed{Z_{\gamma_i} - Z_{\gamma_j} = \int_{\gamma_{ij}} \lambda}$$

γ_{ij} Homology of an ***open*** path ***on Σ*** joining x_i to x_j in the fiber over z.

$$\gamma_{ij} \in \Gamma_{ij} = H_1(\Sigma, \{x_i, x_j\}; \mathbb{Z})$$

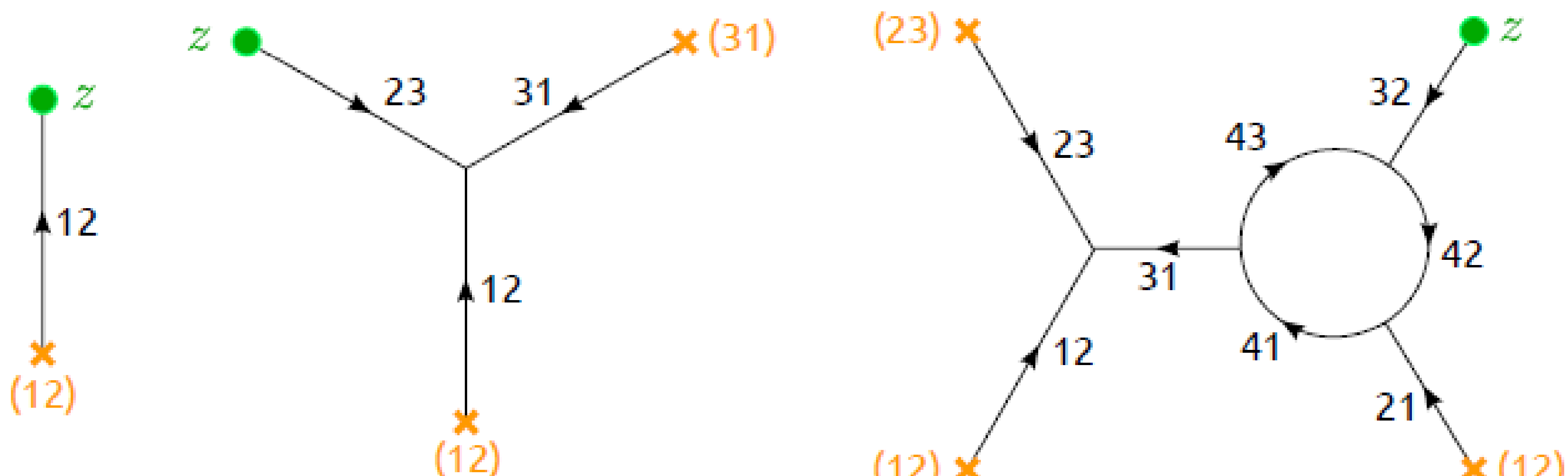
Soliton Charges in Class S



$$\gamma_{ij} \in \Gamma_{ij} = H_1(\Sigma, \{x_i, x_j\}; \mathbb{Z})$$

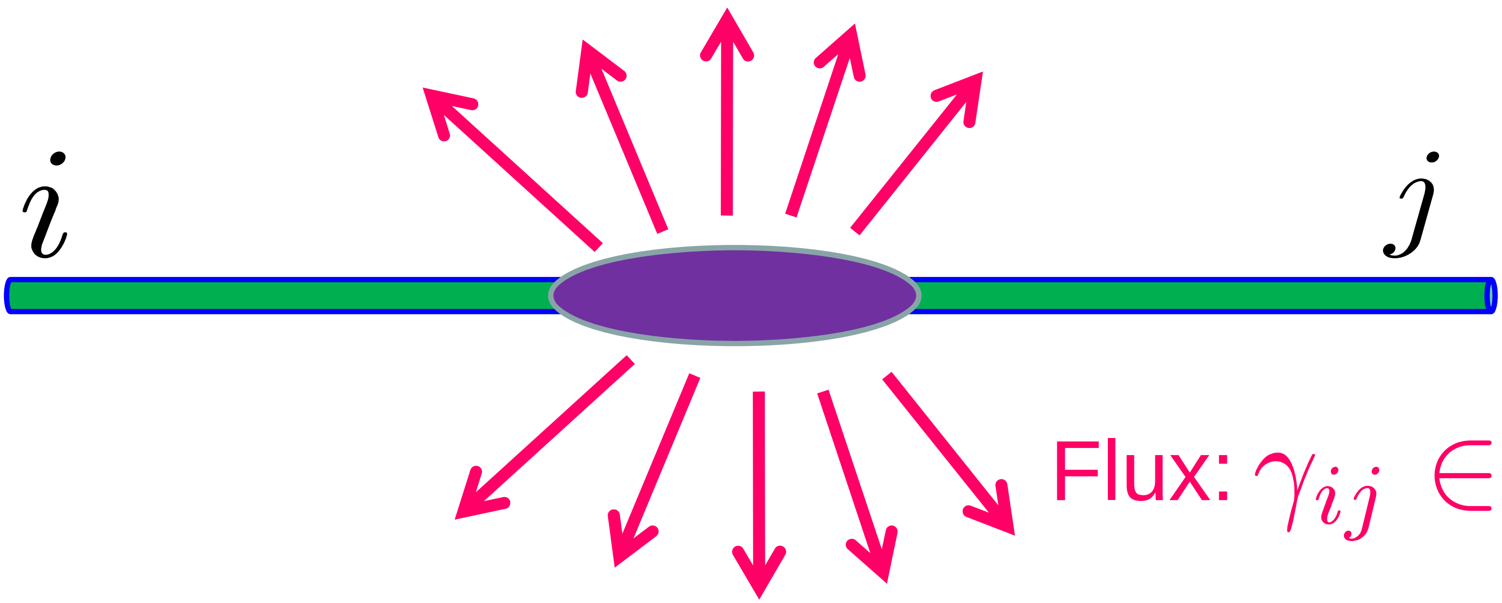
Solitons as open string webs

Solitons for $\mathbb{S}_z$ correspond to open string webs on C which begin and end at z



For solitons on $\mathbb{S}_z$ we define an index μ := signed sum over open string webs beginning and ending at z

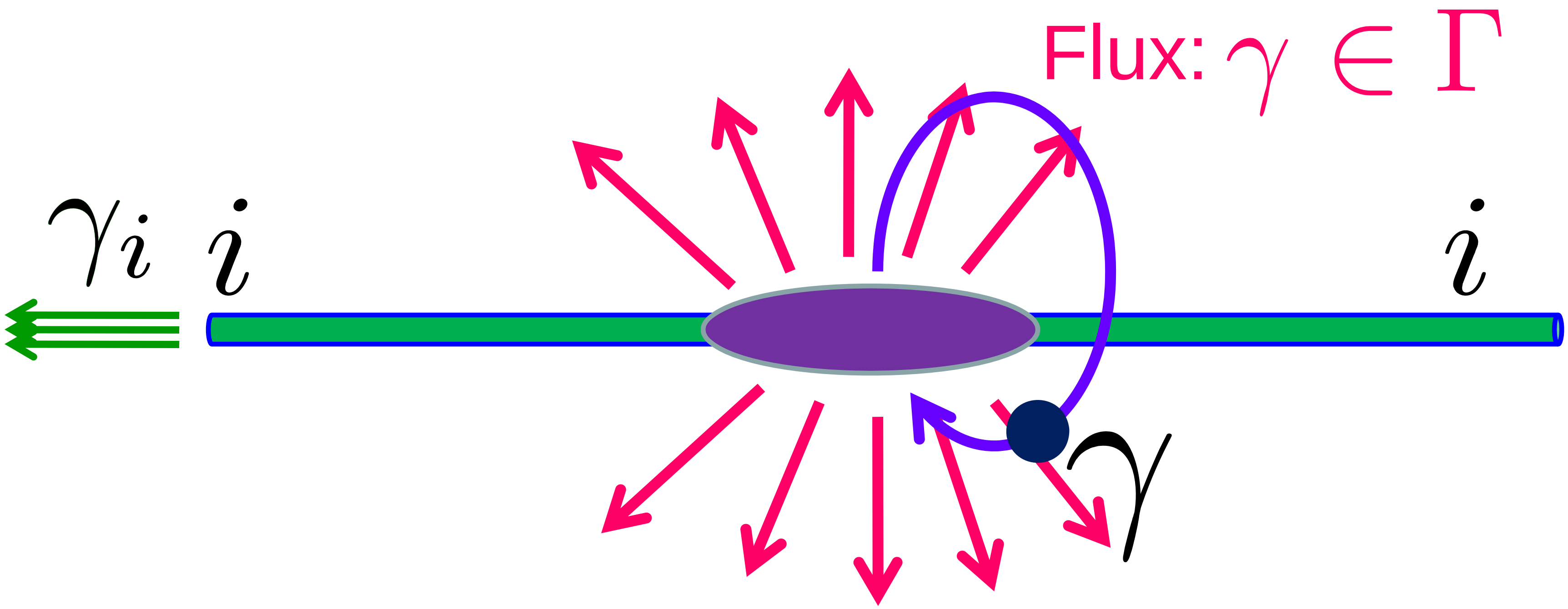
Solitons in Coupled 2d4d Systems



Flux: $\gamma_{ij} \in \Gamma_{ij}$

2D soliton degeneracies: $\mu(\gamma_{ij})$

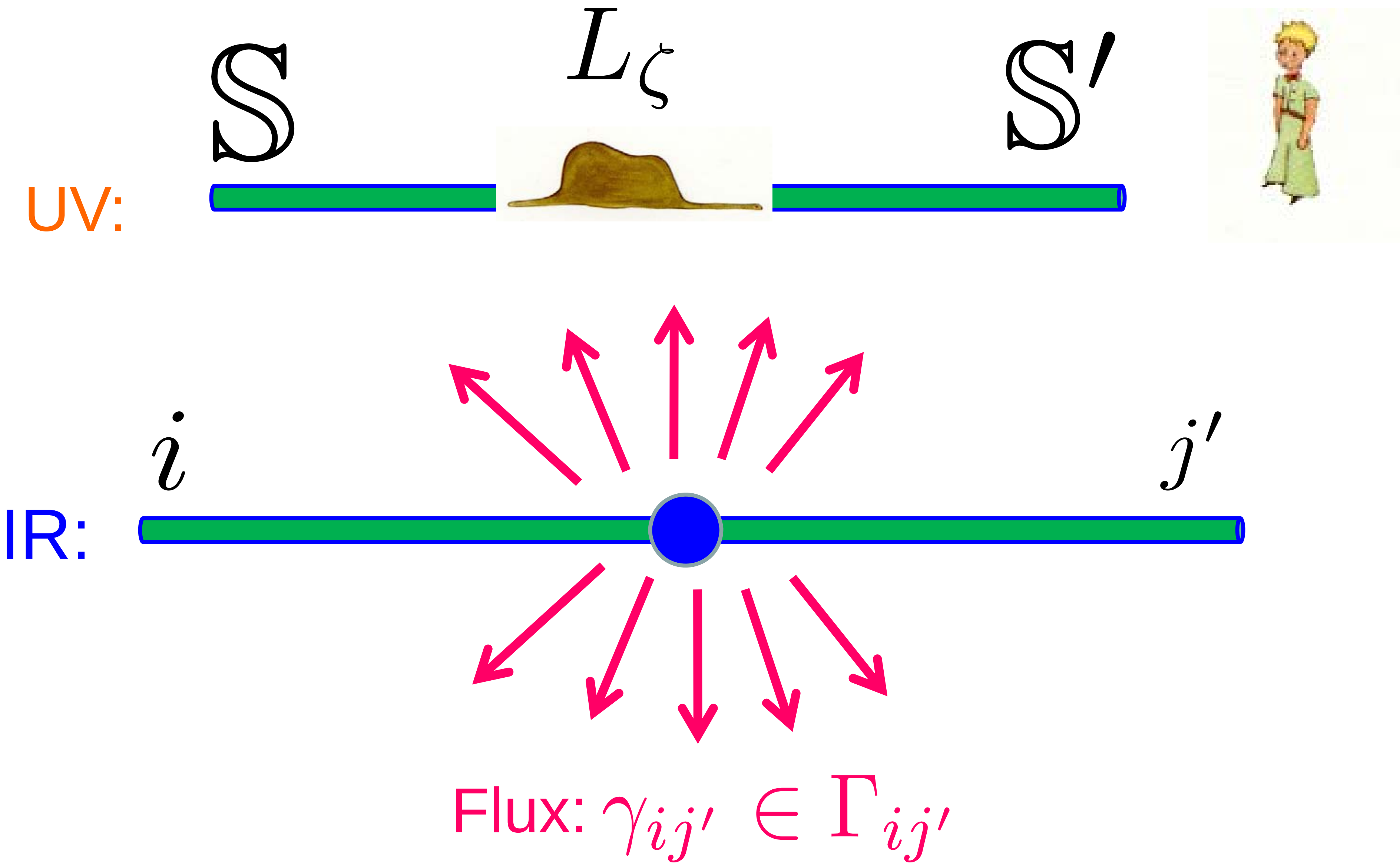
2d/4d Degeneracies: ω



Degeneracy: $\omega(\gamma; \gamma_i)$

$$\omega(\gamma; \gamma_i + \gamma') = \omega(\gamma; \gamma_i) + \Omega(\gamma)\langle \gamma, \gamma' \rangle$$

Supersymmetric Interfaces



Susy Interfaces: Framed Degeneracies

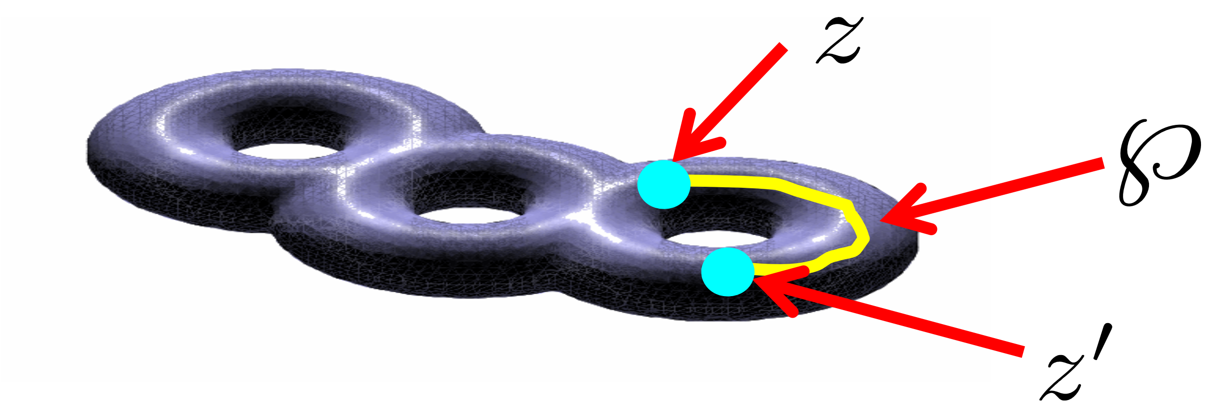
$$\mathcal{H}_{\mathbb{S}L\mathbb{S}'} = \bigoplus_{\gamma_{ij'} \in \Gamma_{ij'}} \mathcal{H}_{\mathbb{S}L\mathbb{S}', \gamma_{ij'}}$$

Our interfaces preserve two susy's of type ζ and hence we can define framed BPS states and form:

$$\underline{\overline{\Omega}}(L, \gamma_{ij'}) = \mathrm{Tr}_{\mathcal{H}_{\mathbb{S}L\mathbb{S}', \gamma_{ij'}}} (-1)^F$$

Susy interfaces for S[$\mathfrak{g}$,C,D]

Interfaces between $\mathbb{S}_z$ and $\mathbb{S}_{z'}$ are labeled by homotopy classes of ***open paths*** $\wp$ on C

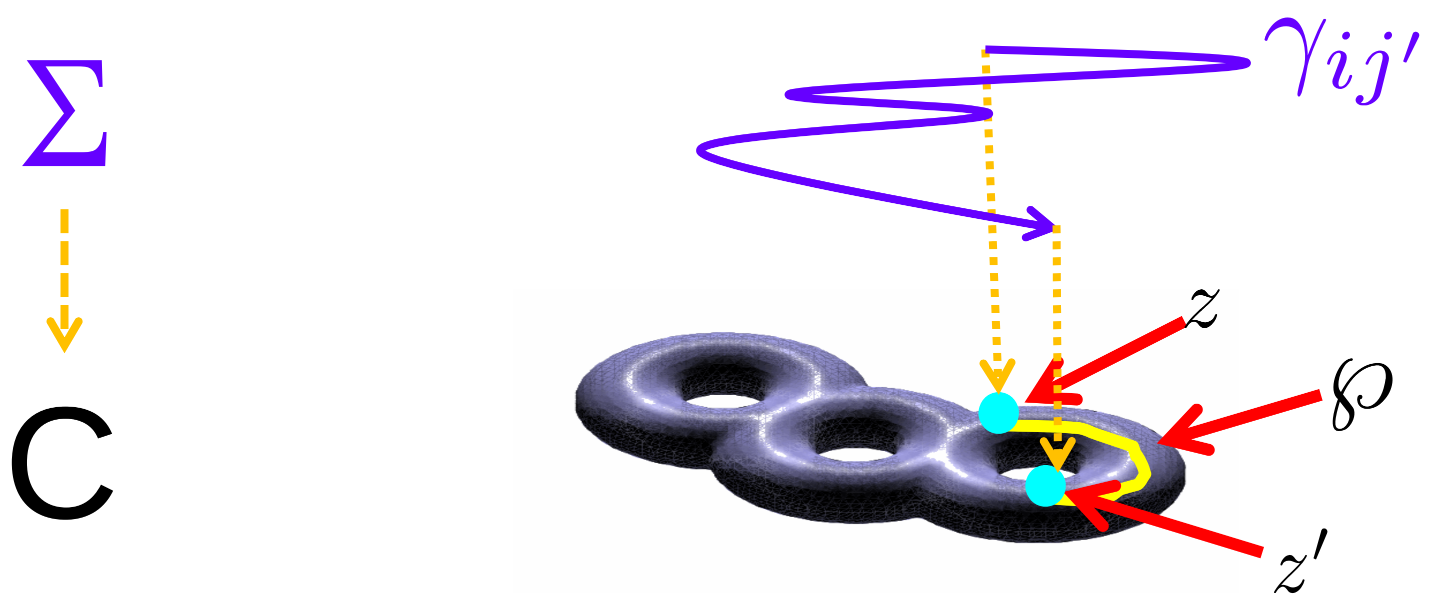


$L_{\wp,\vartheta}$ only depends on the homotopy class of $\wp$

IR Charges of framed BPS

Framed BPS states are graded by homology of open paths $\gamma_{ij'}$ on Σ with endpoints over z and z'

$$\Gamma_{ij'} = H_1(\Sigma, \{x_i, x_{j'}\}; \mathbb{Z})$$



SUMMARY SLIDE

FIELD THEORY	BPS DEGENERACY	CLASS S REALIZATION
BPS PARTICLES	$\Omega(\gamma)$	string webs on C lifting to $\gamma \in H_1(\Sigma)$
LINE DEFECT & Framed BPS	$\underline{\overline{\Omega}}(L_{\wp}, \gamma)$	UV:closed $\wp \subset$ C IR: closed $\gamma \in \Sigma$
SURFACE DEFECT & Solitons	$\mu(\gamma_{ij})$ $\omega(\gamma, \gamma_i)$	UV: $\mathbb{S}_z$ IR: Open paths on Σ joining sheets i and j above z.
SUSY INTERFACE	$\underline{\overline{\Omega}}(L_{\wp}, \gamma_{ij'})$	UV: Open path $\wp$ on C z to z' IR: Open path on Σ from x_i to $x_{j'}$

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Spectral Networks

We will now show how the technique of spectral networks allows us to compute all these BPS degeneracies.

Fix ϑ. The spectral network $\mathcal{W}_\vartheta$ is the collection of points on C given by those z $\in$ C so that there is some 2d soliton on $\mathbb{S}_z$ of phase $\zeta=e^{i\vartheta}$:

$$\mu(\gamma_{ij}) \neq 0 \qquad\qquad \zeta^{-1} Z_{\gamma_{ij}} \in \mathbb{R}_-$$

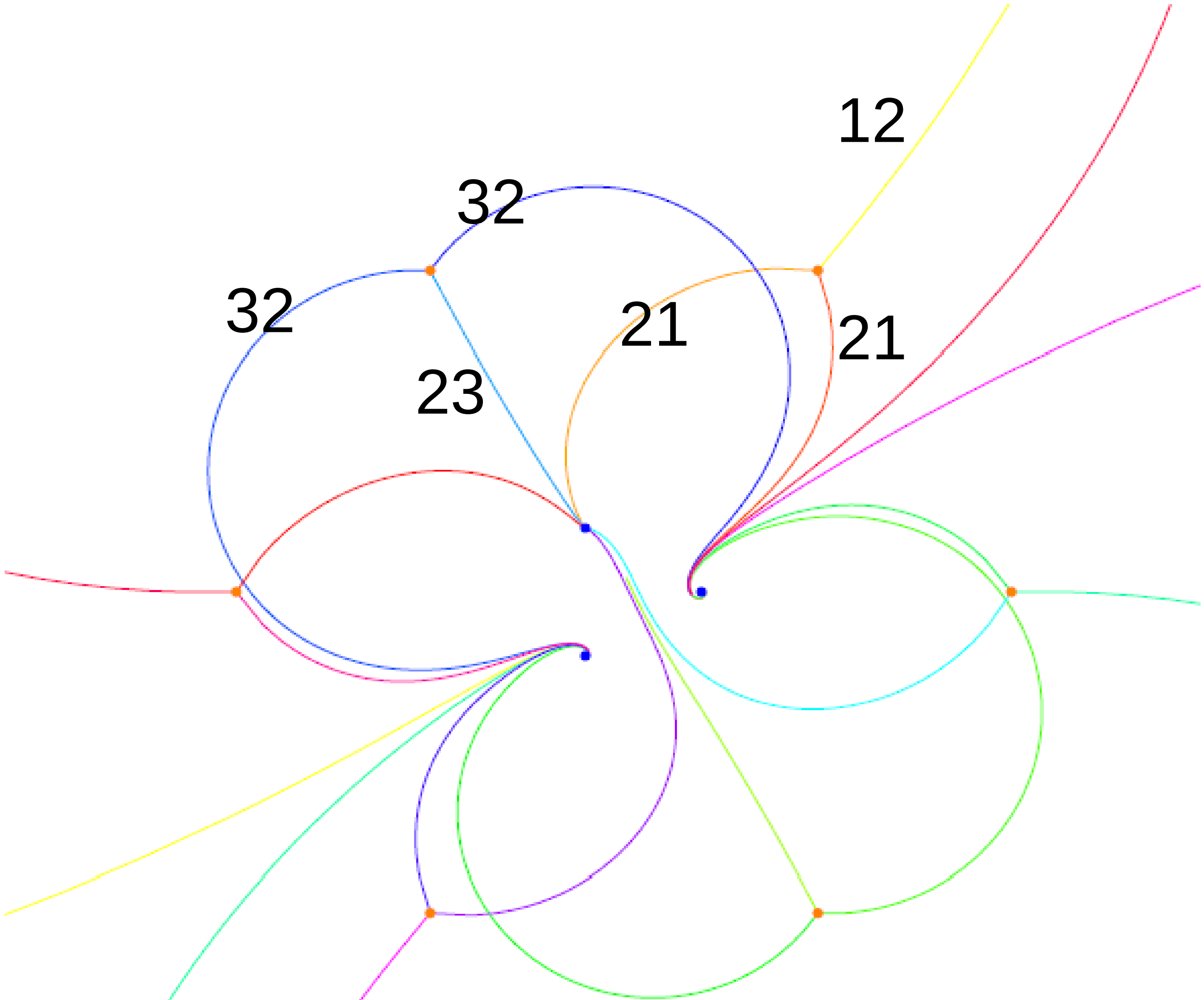
S-Walls

$\mathcal{W}_{\vartheta}$ contains the endpoints z of open string webs of phase ϑ

These webs are made of WKB paths:

$$\langle \lambda_i - \lambda_j , \partial_t \rangle = e^{i\vartheta}$$

The path segments are ``***S-walls of type ij***''

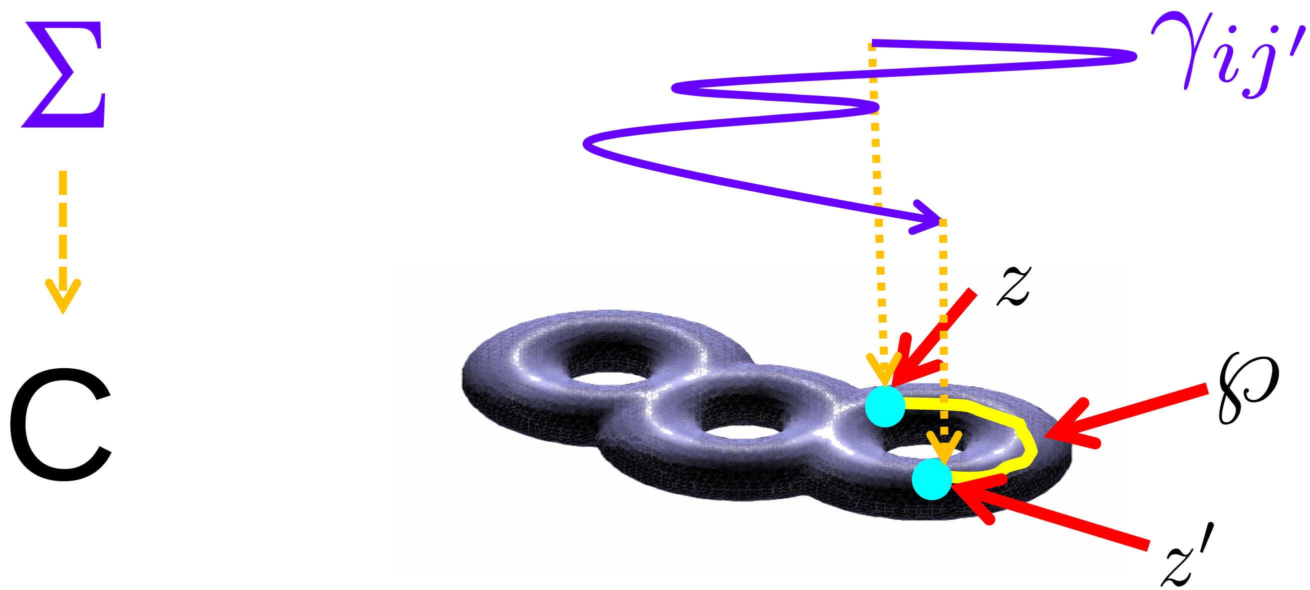


But how do we choose which WKB paths to fit together?

Formal Parallel Transport

Introduce the generating function of framed BPS degeneracies:

$$F(\wp, \vartheta) := \sum_{\Gamma_{ij'}} \overline{\underline{\Omega}}(L_{\wp,\vartheta}, \gamma_{ij'}) X_{\gamma_{ij'}}$$



Homology Path Algebra

To any relative homology class $a \in H_1(\Sigma,\{x_i, x_{j'}\}; \mathbb{Z})$ assign X_a

$$X_a X_b := \begin{cases} X_{a+b} & a, b \text{ composable} \\ 0 & \text{else} \end{cases}$$

X_a generate the "homology path algebra" of Σ

Four Defining Properties of F

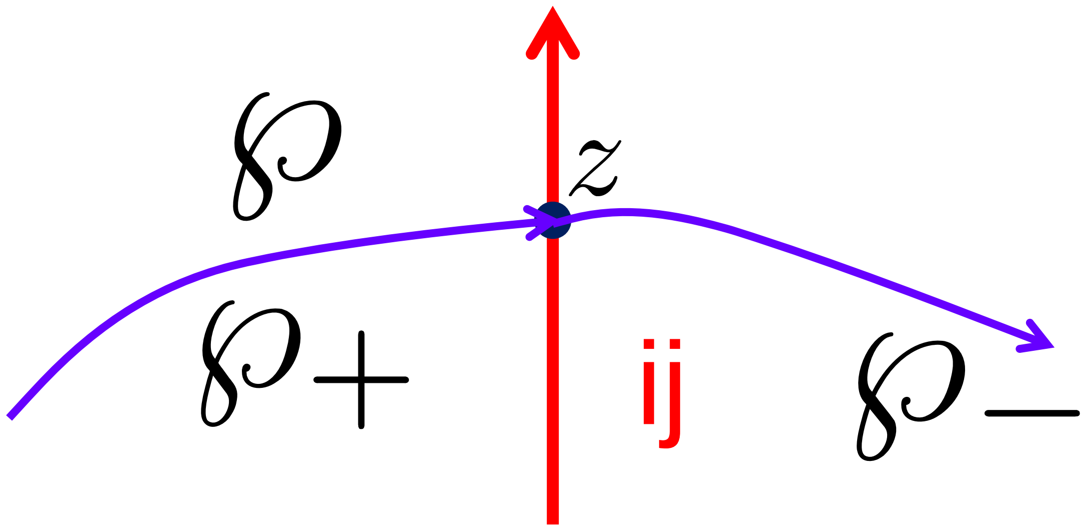
1 $$F(\wp,\vartheta)F(\wp',\vartheta) = F(\wp\wp',\vartheta)$$

2 Homotopy invariance $$F(\wp_1,\vartheta) = F(\wp_2,\vartheta)$$

3 If $\wp$ does NOT intersect $\mathcal{W}_\vartheta$: $$F(\wp,\vartheta) = \sum_{i=1}^{K} X_{\wp^{(i)}}$$

4 If $\wp$ DOES intersect $\mathcal{W}_\vartheta$: ``Wall crossing formula"

Wall Crossing for F($\wp$,ϑ)



$$F(\wp,\vartheta) = \sum_{s=1}^{K} X_{\wp^{(s)}}$$

$$+\mu(\gamma_{ij}) X_{\wp_+^{(i)}} X_{\gamma_{ij}} X_{\wp_-^{(j)}}$$

Theorem: These four conditions completely determine both F($\wp$,ϑ) and μ

Proof:

The mass of a soliton with charge γ_{ij}

$$M(z) = \int_{\gamma_{ij}(z)} e^{-i\vartheta}(\lambda_i - \lambda_j)$$

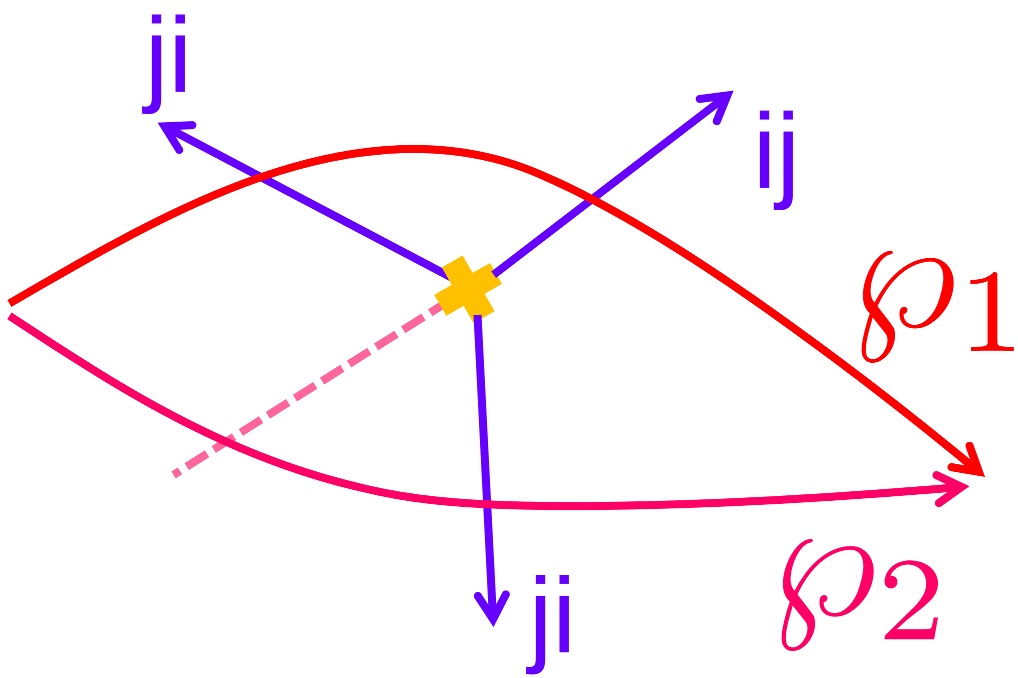
increases monotonically along the S-walls.

Natural mass filtration defines $\mathcal{W}[\Lambda]$:

$$|\zeta^{-1} Z_{\gamma_{ij}}| \le \Lambda$$

Evolving the network -1/3

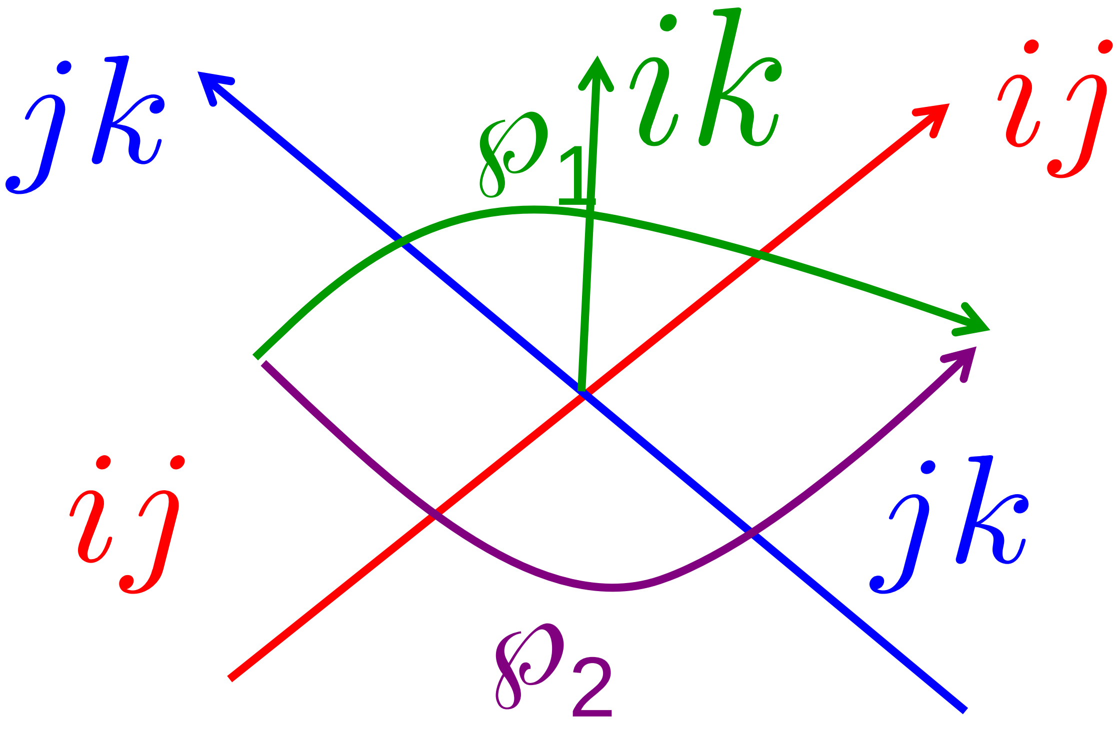
For small Λ the network simply consists of 3 trajectories emitted from each ij branch point,



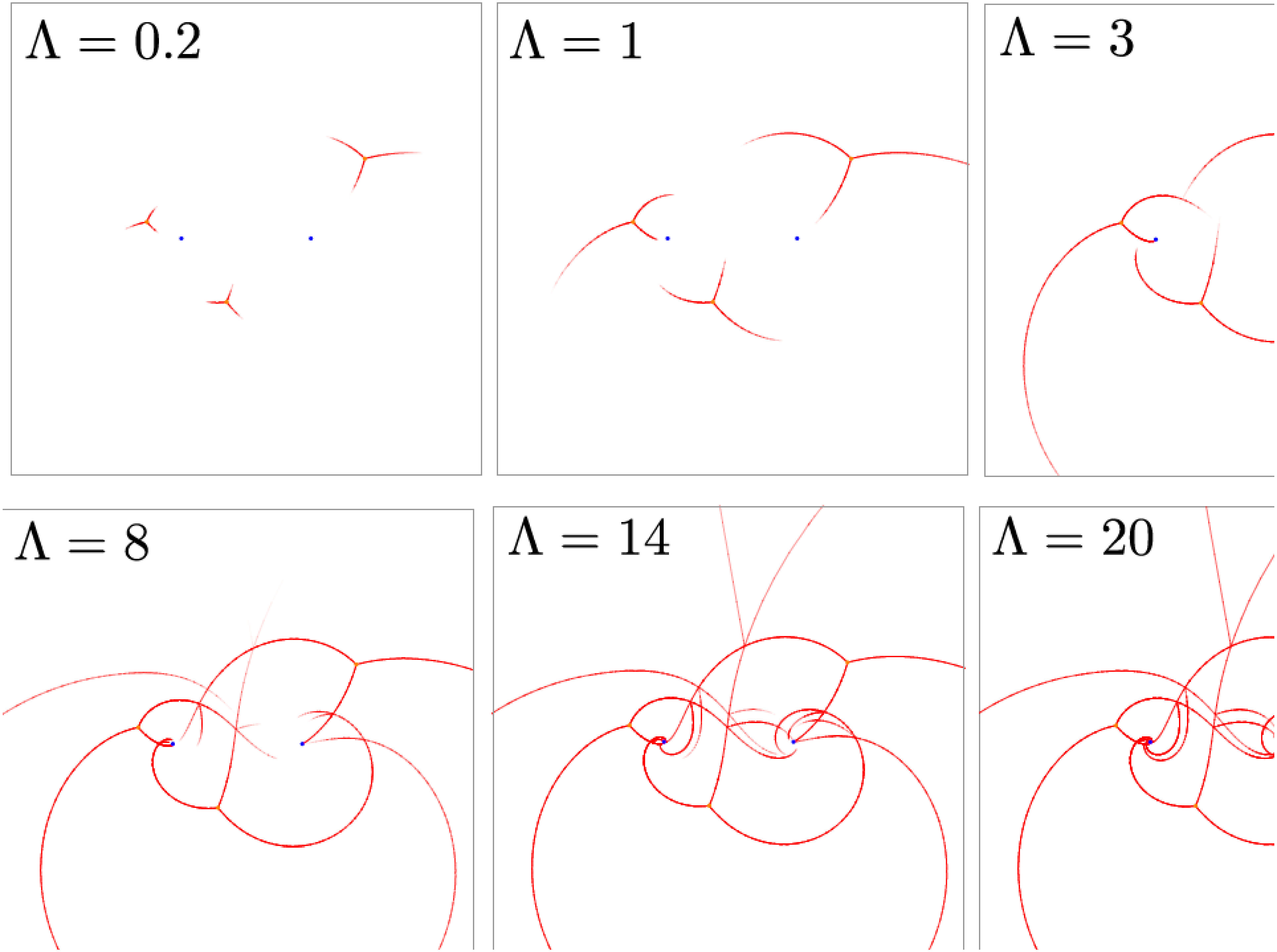
Homotopy invariance implies $\mu(\gamma_{ij})$=1

Evolving the network -2/3

As we increase Λ some trajectories will intersect. The further evolution is again determined by homotopy invariance



and,
$\mu(\gamma_{ik})$ is completely determined (CVWCF)


$\Lambda = 0.2$
$\Lambda = 1$
$\Lambda = 3$
$\Lambda = 8$
$\Lambda = 14$
$\Lambda = 20$

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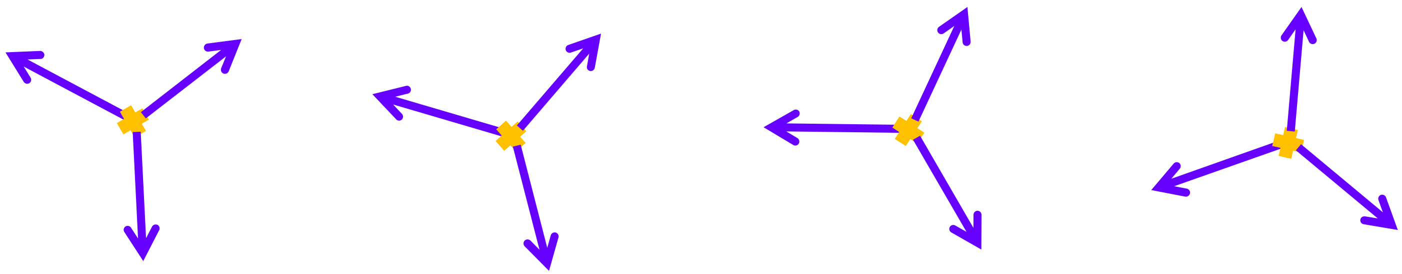
Determine the 2d spectrum

Now vary the phase ζ:

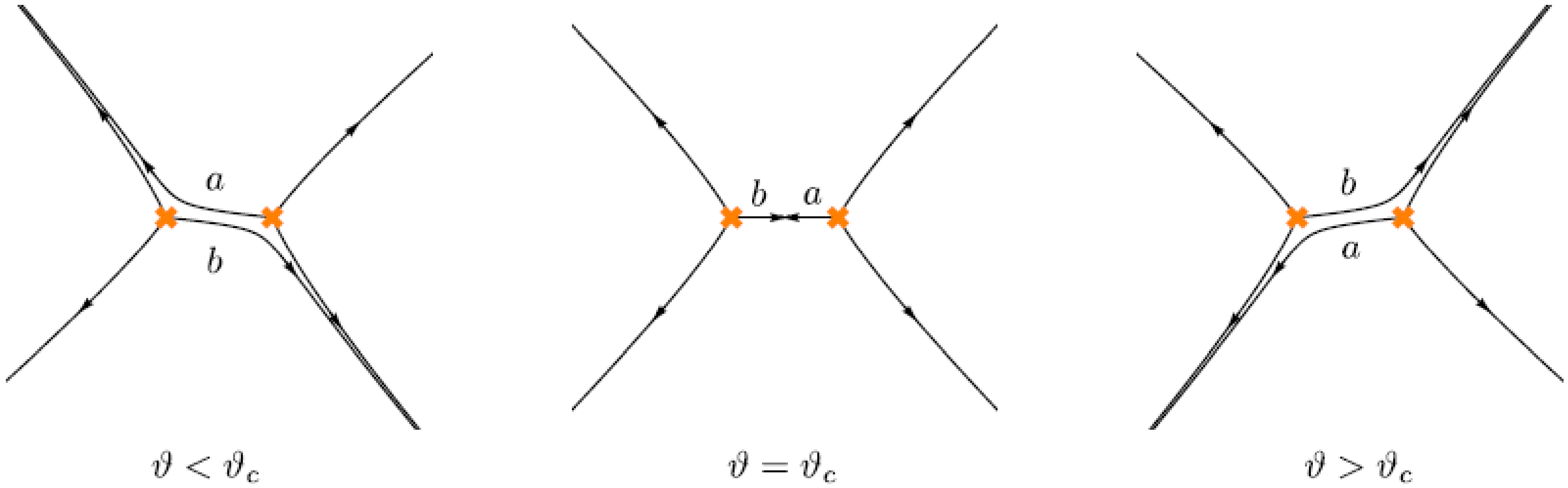


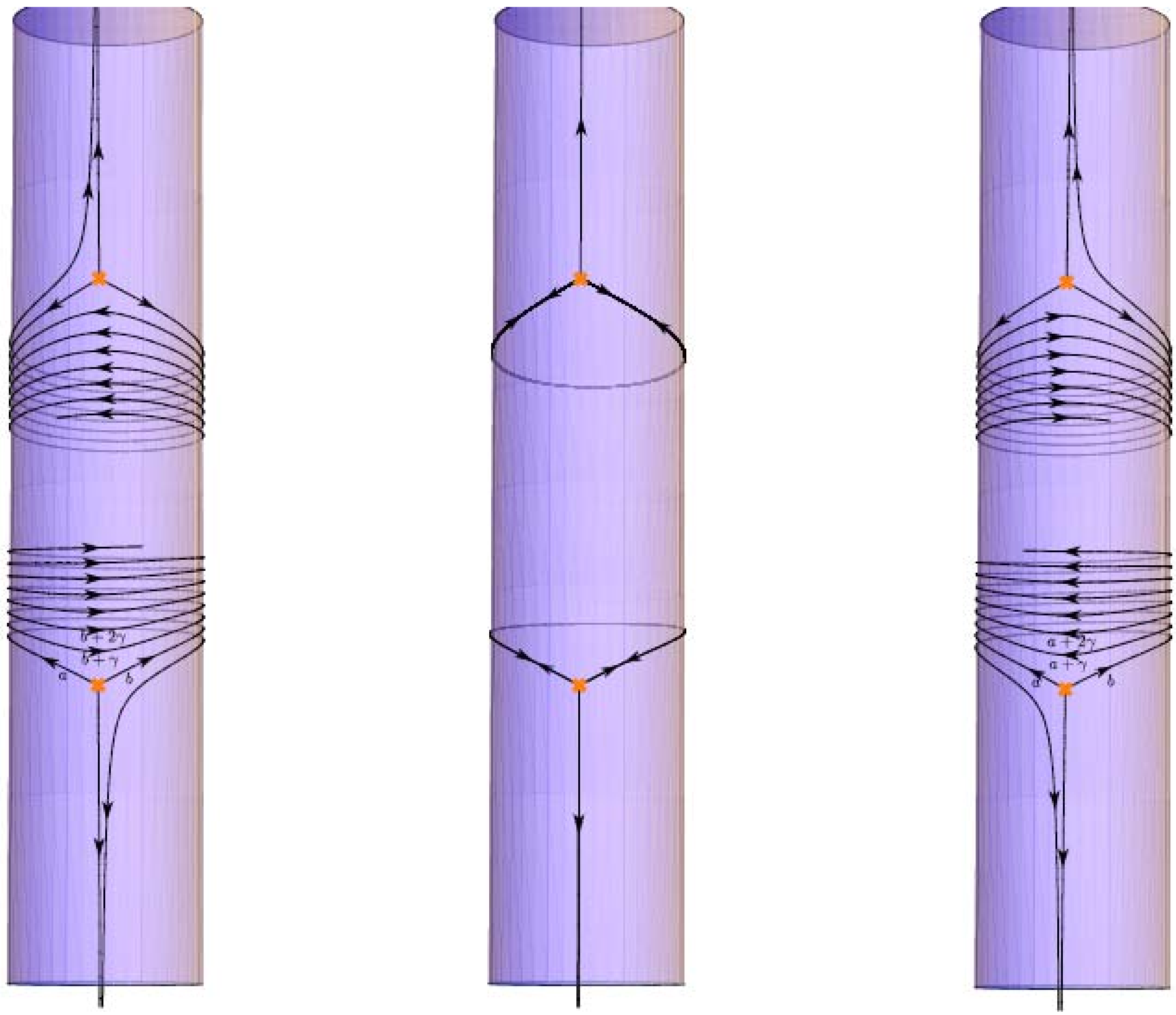
This determines the entire 2d spectrum:

$$\mu(\gamma_{ij}) \quad \text{for all} \quad \mathbb{S}_z,\ i,\ j$$

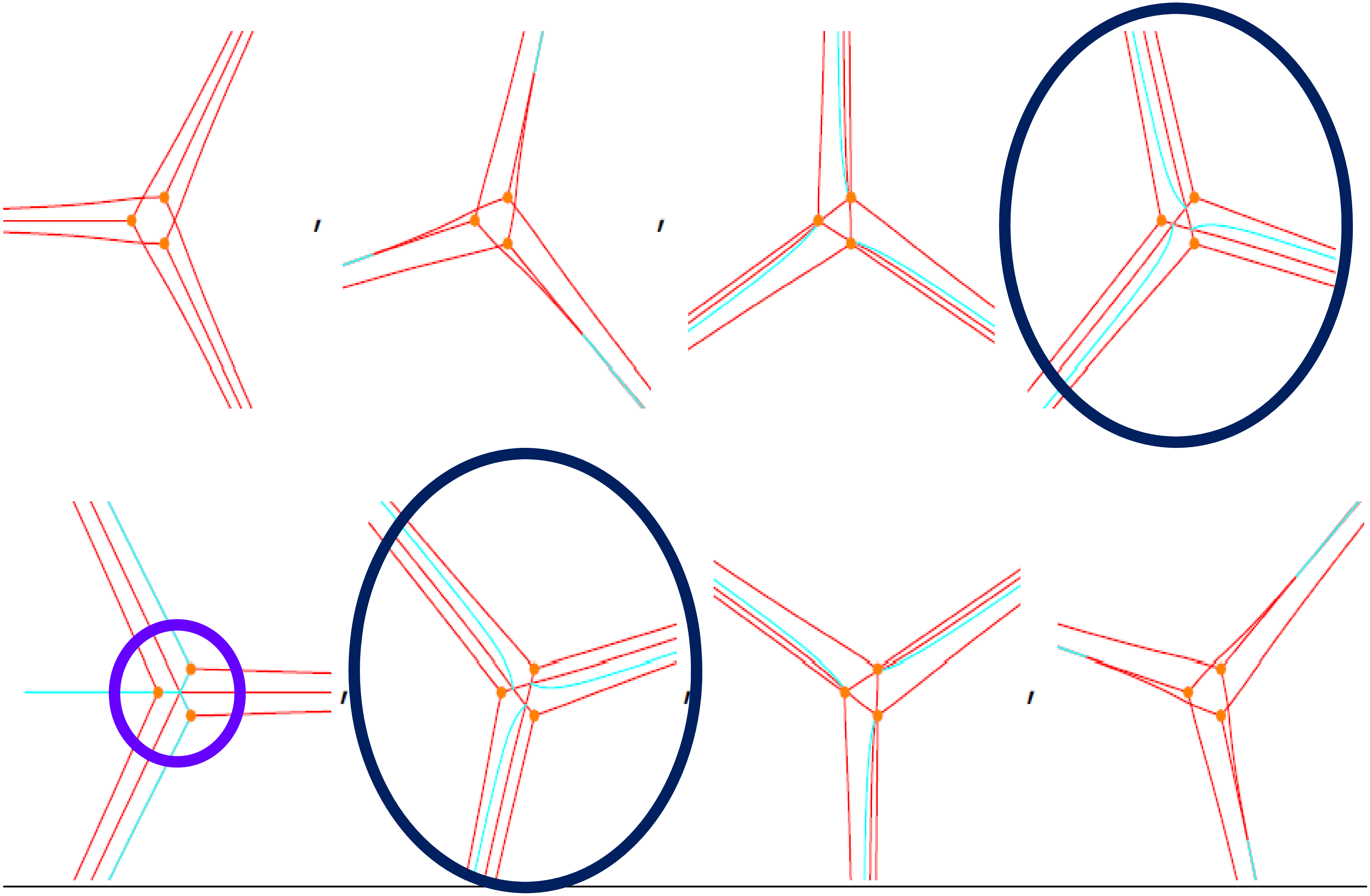
But also the spectral network changes discontinuously for phases ϑ_{c} of ζ corresponding to 4d BPS states!

Movies:
http://www.ma.utexas.edu/users/neitzke/movies/

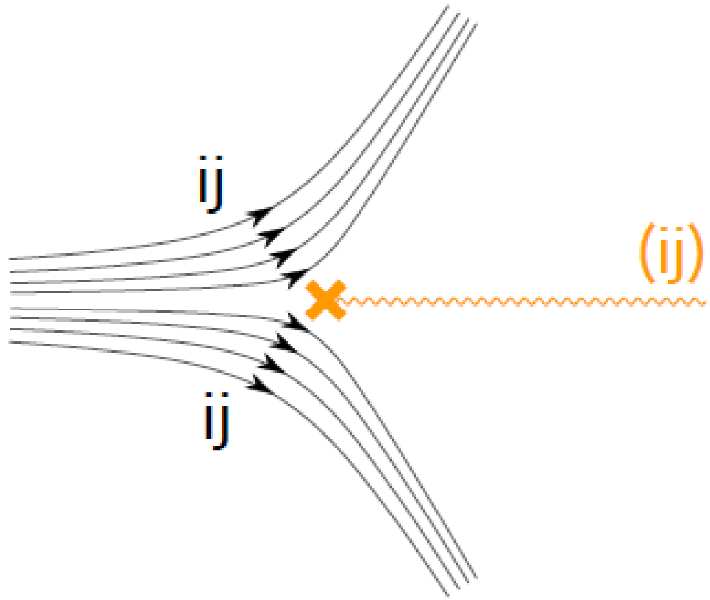




$\vartheta < \vartheta_c$ $\vartheta = \vartheta_c$ $\vartheta > \vartheta_c$



How does a spectral network jump discontinuously?



An ij S-wall crashes into an (ij) branch point

This happens precisely when there are string webs!

So ϑ_c is the phase of a charge of a ***<u>4d BPS state</u>***

Explicit Formula for ω

$$F(\wp, \vartheta_c^{+}) = K F(\wp, \vartheta_c^{-})$$

$$K(X_a) = \prod_{n=1}^{\infty} (1 + \xi_n X_{n\gamma})^{\langle a, L(n\gamma)\rangle} X_a$$

L(nγ) is explicitly constructible from the spectral network.

$$\xi_n = \pm 1$$

$$\omega(n\gamma, a) = \langle L(n\gamma), a\rangle$$

The 2D spectrum

determines

the 4D spectrum.

Spin Lifts

This technique is especially effective in a nice corner of the Coulomb branch of some su(k) theories

Consider an su(2) spectral curve:

$$\det\left(\lambda \mathbf{1}_{2\times 2} - \varphi\right) = 0$$

T_j := Spin j rep. of sl(2)

$$\det\left(\lambda \mathbf{1}_{k\times k} - T_j(\varphi)\right) = 0$$

k = 2j +1

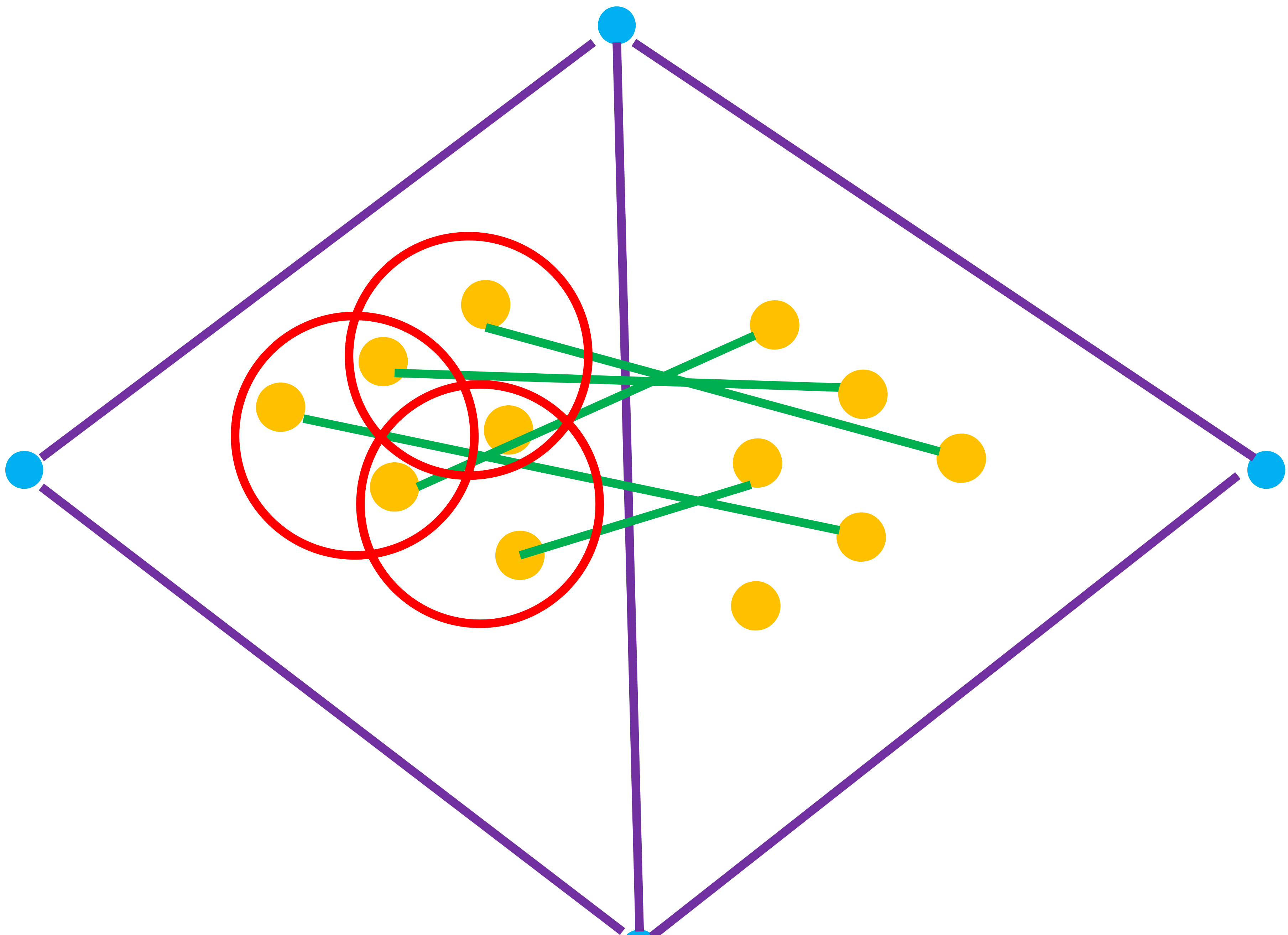
Spin Lifts - B

$$\det\left(\lambda \mathbf{1}_{k\times k} - T_j(\varphi)\right) = 0$$

is a degenerate su(k) spectral curve

Small perturbations deform it to a smooth SW curve of an su(k) theory

Our algorithm gives the BPS spectrum of this su(k) theory in this neighborhood of the Coulomb branch.



Outline

- Introduction
- Theories of class S & their BPS states
- Line defects and framed BPS states
- Surface defects & susy interfaces
- Spectral networks
- Determining the BPS degeneracies
- Conclusion

Conclusion

We introduced "spectral networks," a new combinatorial tool in supersymmetric field theory related to the physics of line and surface defects in N=2 theories of class S.

What are they good for?

as I already said….

They determine BPS degeneracies in 4D N=2 field theories of class S.

They give a "pushforward map" from flat U(1) gauge fields on Σ to flat nonabelian gauge fields on C.

They determine cluster coordinates on the moduli space of flat GL(K,$\mathbb{C}$) connections over C.

"Fock-Goncharov coordinates" "Higher Teichmuller theory"

Higher rank WKB theory

Future Applications?

1. Geometric Langlands program??

2. Knot categorification?

3. Explicit hyperkahler metrics?

Especially, K3.